IRREDUCIBLE REPRESENTATIONS OF C*-CROSSED PRODUCTS BY FINITE GROUPS

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Abstract. We describe the structure of the irreducible representations of crossed products of unital C*-algebras by actions of finite groups in terms of irreducible representations of the C*-algebras on which the groups act. We then apply this description to derive a characterization of irreducible representations of crossed-products by finite cyclic groups in terms of representations of the C*-algebra and its fixed point subalgebra. These results are applied to crossed-products by the permutation group on three elements and illustrated by various examples.

1. Introduction

What is the structure of irreducible representations of C*-crossed-products $A \rtimes_\alpha G$ of an action α of a finite group G on a unital C*-algebra A? Actions by finite groups provide interesting examples, such as quantum spheres [1, 2] and actions on the free group C*-algebras [3], among many examples, and have interesting general properties, as those found for instance in [9]. Thus, understanding the irreducible representations of their crossed-products is a natural inquiry, which we undertake in this paper.

After we wrote this paper, we are shown [11] where Takesaki provides in a detailed description of the irreducible representations of $A \rtimes G$ when G is a locally group acting on a type I C*-algebra A and the action is assumed *smooth*, as defined in [11, Section 6]. Our paper takes a different road, though with some important intersections we were not aware of originally. Both our paper and [11] make use of the Mackey machinery and the structure of the commutant of the image of

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irreducible representations of the crossed-products. However, since our C*-algebras are not assumed to be type I, and in general the restriction of an irreducible representation of $A \rtimes G$ to A does not lead to an irreducible representation of A, we need a different approach than [11]. The main tool we use for this purpose is the impressive result proven in [7] that for ergodic actions of compact groups on unital C*-algebras, spectral subspaces are finite dimensional. As a consequence, we can analyze irreducible representations of finite group crossed-products on arbitrary unital C*-algebras with no condition on the action of the group on the spectrum of the C*-algebra. More formally, we restrict the assumption on the group and relax it completely on the C*-algebra and the action compared to [11, Theorem 7.2].

Our research on this topic was initiated in a paper of Choi and the second author [4] in the case where $G = \mathbb{Z}_2$, i.e. for the action of an order two automorphism σ on a C*-algebra A. In this situation, all irreducible representations of $A \rtimes_\sigma \mathbb{Z}_2$ are either *minimal*, in the sense that their restriction to A is already irreducible, or are regular, i.e. induced by a single irreducible representation π of A such that π and $\pi \circ \sigma$ are not equivalent. In this paper, we shall answer the question raised at the beginning of this introduction for any finite group G. Thus, we suppose given any action α of G on a unital C*-algebra A. In this general situation, we show that for any irreducible representation Π of $A \rtimes_\alpha G$ on some Hilbert space $\mathcal{H}$, the group G acts ergodically on the commutant $\Pi(A)'$ of $\Pi(A)$, and thus, by a theorem of Hoegh-Krohn, Landstad and Stormer [7], we prove that $\Pi(A)'$ is finite dimensional. We can thus deduce that there is a subgroup H of G such that Π is constructed from an irreducible representation Ψ of $A \rtimes_\alpha H$, with the additional property that the restriction of Ψ to A is the direct sum of finitely many representations all equivalent to an irreducible representation π of A. In addition, the group H is exactly the group of elements h in G such that π and $\pi \circ \alpha_h$ are equivalent. The canonical unitaries of $A \rtimes_\alpha G$ are mapped by Π to generalized permutation operators for some decomposition of $\mathcal{H}$. This main result is the matter of the third section of this paper.

When G is a finite cyclic group, then we show that the representation Ψ is in fact minimal and obtain a full characterization of irreducible representations of $A \rtimes_\alpha G$. This result can not be extended to more generic finite groups, as we illustrate with some examples. In addition, the fixed point C*-subalgebra of A for α plays a very interesting role

in the description of minimal representations when G is cyclic. We investigate the finite cyclic case in the fourth section of this paper.

We then apply our work to the case where G is the permutation group $\mathfrak{S}_3$ on three elements $\{1,2,3\}$. It is possible again to fully describe all irreducible representations of any crossed-product $A \rtimes_\alpha \mathfrak{S}_3$, and we illustrate all the cases we can encounter by examples. This matter is discussed in the last section of this paper.

We start our paper with a section on generalities on crossed-products of C*-algebras by finite groups, including a result on a characterization of irreducible regular representations. This section also allows us to set some of our notations. We now fix some other notations which we will use recurrently in this paper. Given a Hilbert space $\mathcal{H}$ which we decompose as a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_m$ of Hilbert subspaces, we shall write an operator T on $\mathcal{H}$ as an $m \times m$ matrix whose (i,j)-entry is the operator $p_i T p_j$ where $p_1, \ldots, p_m$ are the orthogonal projections from $\mathcal{H}$ onto respectively $\mathcal{H}_1, \ldots, \mathcal{H}_m$. If $t_1, \ldots, t_m$ are operators on, respectively, $\mathcal{H}_1, \ldots, \mathcal{H}_m$, then the diagonal operator with entries $t_1, \ldots, t_m$ will be denoted by $t_1 \oplus \ldots \oplus t_m$, i.e. $\oplus_{j=1}^m t_j \left(\xi_1, \ldots, \xi_m\right) = \left(t_1\xi_1, \ldots, t_m\xi_m\right)$ for all $(\xi_1, \ldots, \xi_i) \in \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_m$. If $\pi_1, \ldots, \pi_m$ are representations of some C*-algebra A acting respectively on $\mathcal{H}_1, \ldots, \mathcal{H}_m$, then the representation $\pi_1 \oplus \ldots \oplus \pi_m$ of A on $\mathcal{H}$ is defined by $\left(\pi_1 \oplus \ldots \oplus \pi_m\right)(a) = \pi_1(a) \oplus \ldots \oplus \pi_m(a)$ for all $a \in A$. The identity operator of $\mathcal{H}$ will be denoted by $1_\mathcal{H}$ or simply 1 when no confusion may occur. More generally, when an operator t on a Hilbert space $\mathcal{H}$ is a scalar multiple $\lambda 1_\mathcal{H}$ ($\lambda \in \mathbb{C}$) of the identity of $\mathcal{H}$ we shall simply denote it by λ and omit the symbol $1_\mathcal{H}$ when appropriate.

We shall denote by $f_{|E_0}$ the restriction of any function $f : E \longrightarrow F$ to a subset E_0 of E. The set $\mathbb{T}$ is the unitary group of $\mathbb{C}$, i.e. the set of complex numbers of modulus 1.

2. Crossed-Product By Finite Groups

In this paper, we let A be a unital C*-algebra and α an action on A of a finite group G by *-automorphisms. A covariant representation of (A, α, G) on a unital C*-algebra B is a pair (π, V) where π is a *-homomorphism from A into B and V is a group homomorphism from G into the unitary group of B such that for all $g \in G$ and $a \in A$ we have $V(g)\pi(a)V\left(g^{-1}\right) = \pi \circ \alpha_g(a)$. The crossed-product C*-algebra $A \rtimes_\alpha G$ is the universal C*-algebra among all the C*-algebras generated by some covariant representation of (A, α, G). In particular, $A \rtimes_\alpha G$

is generated by a copy of A and unitaries U^g for $g \in G$ such that $U^{gh} = U^g U^h$, $U^{g^{-1}} = (U^g)^*$ and $U^g a U^{g^{-1}} = \alpha_g(a)$ for all $g, h \in G$ and $a \in A$. The construction of $A \rtimes_\alpha G$ can be found in [8] and is due originally to [12].

By universality, crossed-products by finite groups have a very simple form which we now describe.

Proposition 2.1. *Let G be a finite group of order n and write $G = \{g_0, \ldots, g_{n-1}\}$ with g_0 the neutral element of G. Let σ be the embedding of G in the permutation group of $\{0, \ldots, n-1\}$ given by $\sigma_g(i) = j$ if and only if $gg_i = g_j$ for all $i, j \in \{0, \ldots, n-1\}$ and $g \in G$. We now define V_g to be the matrix in $M_n(A)$ whose (i,j) entry is given by 1_A if $\sigma_g(i) = j$ and 0 otherwise, i.e. the tensor product of the permutation matrix for σ_g and $1_{M_n}(A)$. Let $\psi : A \longrightarrow M_n(A)$ be the *-monomorphism:*

$$\psi : a \in A \longmapsto \begin{bmatrix} a & & & \\ & \alpha_{g_1}(a) & & \\ & & \ddots & \\ & & & \alpha_{g_{n-1}}(a) \end{bmatrix}.$$

*Then $A \rtimes_\alpha G$ is *-isomorphic to $\oplus_{g \in G} \psi(A) V_g$. In particular:*

$$\bigoplus_{g \in G} AU^g = A \rtimes_\alpha G.$$

Proof. The embedding of G into permutations of G is of course the standard Cayley Theorem. We simply fix our notations more precisely so as to properly define our embedding ψ. A change of indexing of G simply correspond to a permutation of the elements in the diagonal of ψ and we shall work modulo this observation in this proof. For $b \in M_n(A)$ we denote by $b_{i,i'}$ its (i,i')-entry for $i, i' \in \{1, \ldots, n\}$.

An easy computation shows that:

$$V_g \psi(a) V_{g^{-1}} = \psi\left(\alpha_g(a)\right)$$

and $V_g V_h = V_{gh}$ for all $g, h \in G$ and $a \in A$. Therefore, by universality of $A \rtimes_\alpha G$, there exists a (unique) *-epimorphism $\eta : A \rtimes_\alpha G \twoheadrightarrow \oplus_{g \in G} \psi(A) V_g$ such that $\eta_{|A} = \psi$ and $\eta(U^g) = V_g$ for $g \in G$. Our goal is to prove that η is a *-isomorphism.

First, we show that $\oplus_{g \in G} AU^g$ is closed in $A \rtimes_\alpha G$.

Let $(a_m^0,\dots,a_m^{n-1})_{m\in\mathbb{N}}$ in A^n such that $\left(\sum_{j=0}^{n-1} a_m^j U^{g_j}\right)_{m\in\mathbb{N}}$ is a convergent sequence in $A\rtimes_\alpha G$. Now:

$$\eta\left(\sum_{j=0}^{n-1} a_m^j U^{g_j}\right)=\sum_{j=0}^{n-1}\psi(a_m^j)V_g.$$

By definition, we have $\sigma_{g_j}(0)=i$ for all $i\in\{0,\dots,n-1\}$. Let $j\in\{0,\dots,n-1\}$. Then $V_{j+1,1}^{g_j}=1_A$ and $V_{j+1,1}^{g_i}=0$ for all $i\in\{0,\dots,n-1\}\setminus\{j\}$. Hence, $\left(\eta\left(\sum_{j=1}^{n} a_m^j U^{g_j}\right)\right)_{1,j+1}=a_m^j$ for all $m\in\mathbb{N}$. Since η is continuous, and so is the canonical projection $b\in M_n(A)\longrightarrow b_{1,j+1}\in A$, we conclude that $(a_m^j)_{m\in\mathbb{N}}$ converges in A. Let $a^j\in A$ be its limit. Then $(a_m^0,\dots,a_m^{n-1})_{m\in\mathbb{N}}$ converges in A^n to $(a^0,\dots,a^{n-1})$. Thus, $\left(\sum_{j=0}^{n-1} a_m^j U^{g_j}\right)_{m\in\mathbb{N}}$ converges to $\sum_{j=0}^{n-1} a^j U^{g_j}\in\oplus_{g\in G}AU^g$ and thus $\oplus_{g\in G}AU^g$ is closed in $A\rtimes_\alpha G$. Since $\oplus_{g\in G}AU^g$ is dense in $A\rtimes_\alpha G$ by construction, we conclude that $A\rtimes_\alpha G=\oplus_{g\in G}AU^g$.

Now, we show that η is injective. Let $c\in A\rtimes_\alpha G$ such that $\eta(c)=0$. Then there exists $a_0,\dots,a_{n-1}\in A$ such that $c=\sum_{j=0}^{n-1}a_jU^{g_j}$. Let $j\in\{0,\dots,n-1\}$. Then $\eta(c)=0$ implies that $\eta(c)_{j+1,1}=a_j=0$ for all $j\in\{0,\dots,n-1\}$ and thus $c=0$. So η is a *-isomorphism and our proof is concluded. $\square$

As we will focus our attention on the crossed-products by finite cyclic groups in the fourth section of this paper and Proposition (2.1) is particularly explicit in this case, we include the following corollary:

Corollary 2.2. *Let σ be an automorphism of order n of a unital C^*-algebra A. Then $A\rtimes_\sigma\mathbb{Z}_n$ is *-isomorphic to:*

$$\left\{\begin{array}{c}\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n\\ \sigma(a_n) & \sigma(a_1) & \sigma(a_2) & \sigma(a_3) & \\ \sigma^2(a_{n-1}) & \sigma^2(a_n) & \sigma^2(a_1) & \ddots & \ddots\\ \vdots & \ddots & \ddots & \ddots & \sigma^{n-2}(a_2)\\ \sigma^{n-1}(a_2) & \sigma^{n-1}(a_3) & \cdots & \sigma^{n-1}(a_n) & \sigma^{n-1}(a_1)\end{bmatrix}\in M_n(A)\\ a_1,\dots,a_n\in A\end{array}\right\}$$

where U^1 *mapped to* $\begin{bmatrix} 0 & 1 & & 0 \\ \vdots & 0 & \ddots & \\ 0 & \vdots & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$ *and* A *is embedded diagonally as* $a \in A \mapsto \begin{bmatrix} a & & & \\ & \sigma(a) & & \\ & & \ddots & \\ & & & \sigma^{n-1}(a) \end{bmatrix}$. *In particular,* $A \rtimes_\sigma \mathbb{Z}_n = A \oplus AU^1 \oplus \ldots \oplus AU^{n-1}$.

Proof. Simply write $\mathbb{Z}_n = \{0, \ldots, n-1\}$ so that:

$$V_1 = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & 0 & \ddots & \\ 0 & \vdots & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

The result is a direct computation of $\oplus_{k=0}^{n-1} \psi(A) \left(V_1\right)^k$. □

We now turn our attention to the irreducible representations of $A \rtimes_\alpha G$. Proposition (2.1) suggests that we construct some representations from one representation of A and the left regular representation of G. Of particular interest is to decide when such representations are irreducible. We will use many times the following lemma [6, 2.3.4 p. 30], whose proof is included for the reader's convenience:

Lemma 2.3 (Schur)**.** *Let* π_1 *and* π_2 *be two irreducible representations of a* C^**-algebra* A *acting respectively on Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$. *Then* π_1 *and* π_2 *are unitarily equivalent if and only if there exists a nonzero operator* $T : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ *such that for all* $a \in A$ *we have* $T\pi_1(a) = \pi_2(a)T$. *Moreover, if there exists such a nonzero intertwining operator, then it is unique up to a nonzero scalar multiple.*

Proof. If π_1 and π_2 are unitarily equivalent then there exists a unitary T such that for all $a \in A$ we have $T\pi_1(a) = \pi_2(a)T$. In particular, $T \neq 0$. Moreover, assume that there exists T' such that $T'\pi_1 = \pi_2 T'$. Then $T^*T'\pi_1 = T^*\pi_2 T' = \pi_1 T^*T'$. Hence since π_1 is irreducible, there exists $\lambda \in \mathbb{C}$ such that $T' = \lambda T$.

Conversely, assume that there exists a nonzero operator $T : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that for all $a \in A$ we have:

$$T\pi_1(a) = \pi_2(a)T. \tag{2.1}$$

Then for all $a \in A$:

$$T^*T\pi_1(a) = T^*\pi_2(a)T.$$

In particular $T^*T\pi_1(a^*) = T^*\pi_2(a^*)T$ for all $a \in A$. Applying the adjoint operation to this equality leads to $\pi_1(a)T^*T = T^*\pi_2(a)T$ and thus:

$$T^*T\pi_1(a) = \pi_1(a)T^*T.$$

Since π_1 is irreducible, there exists $\lambda \in \mathbb{C}$ such that $T^*T = \lambda 1_{\mathcal{H}_2}$. Since $T \neq 0$ we have $\lambda \neq 0$. Up to replacing T by $\frac{1}{\mu}T$ where $\mu^2 = |\lambda|$ and $\mu \in \mathbb{R}$ we thus get $T^*T = 1_{\mathcal{H}_2}$. Thus T is an isometry. In particular, TT^* is a nonzero projection.

Similarly, we get $\pi_2(a)TT^* = TT^*\pi_2(a)$ and thus TT^* is scalar as well. Hence TT^* is the identity again (As the only nonzero scalar projection) and thus T is a unitary operator. Hence by (2.1), π_1 and π_2 are unitarily equivalent. □

Given a Hilbert space $\mathcal{H}$, the C*-algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$.

Theorem 2.4. *Let G be a finite group with neutral element e and α an action of G on a unital C*-algebra A. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation of A and let λ be the left regular representation of G on $\ell_2(G)$. Let δ_g be the function in $\ell_2(G)$ which is 1 at $g \in G$ and 0 otherwise. Define $\Pi : A \rtimes_\alpha G \to \mathcal{B}(\ell_2(G) \otimes \mathcal{H})$ by*

$$\begin{aligned} \Pi(a)(\delta_g \otimes \xi) &= \delta_g \otimes \pi(\alpha_{g^{-1}}(a))\xi, \text{ and} \\ \Pi(g) &= \lambda(g) \otimes 1_{\mathcal{H}}. \end{aligned}$$

Then Π is irreducible if and only if π is irreducible and π is not unitarily equivalent to $\pi \circ \alpha_g$ for any $g \in G \setminus \{e\}$.

Proof. Assume now that π is irreducible and not unitarily equivalent to $\pi \circ \alpha_g$ whenever $g \in G \setminus \{e\}$. Suppose that Π is reducible. Then there exists a non-scalar operator Ω in the commutant of $\Pi(A \rtimes_\alpha G)$. Now, we observe that the commutant of $\{\lambda(g) \otimes 1_{\mathcal{H}} : g \in G\}$ is $\rho(G) \otimes \mathcal{B}(\mathcal{H})$, where ρ is the right regular representation of G. Hence, there exist an operator T_g on $\mathcal{H}$ for all $g \in G$ such that $\Omega = \sum_{g\in G} \rho(g) \otimes T_g$. For every $\xi \in \mathcal{H}$ and $a \in A$, we have

$$\begin{aligned} \left(\sum_{g\in G} \rho(g) \otimes T_g\right) \Pi(a)(\delta_0 \otimes \xi) &= \left(\sum_{g\in G} \rho(g) \otimes T_g\right)(\delta_0 \otimes \pi(a)\xi) \\ &= \sum_{g\in G} \delta_g \otimes T_g \pi(a)\xi \end{aligned}$$

and

$$\begin{aligned}\Pi(a)\left(\sum_{g\in G}\rho(g)\otimes T_g\right)(\delta_0\otimes\xi) &= \Pi(a)\left(\sum_{g\in G}\delta_g\otimes T_g\xi\right)\\ &= \sum_{g\in G}\delta_g\otimes\pi\left(\alpha_{g^{-1}}(a)\right)T_g\xi.\end{aligned}$$

Therefore, for every $g\in G$ and for all $a\in A$:

$$\pi\left(\alpha_{g^{-1}}(a)\right)T_g = T_g\pi(a)\,. \tag{2.2}$$

Since Ω is non scalar, there exists $g_0\in G\setminus\{e\}$ such that $T_{g_0}\neq 0$. By Lemma (2.3), Equality (2.2) for g_0 implies that π and $\pi\circ\alpha_{g_0}$, which are irreducible, are also unitarily equivalent since $T_{g_0}\neq 0$. This is a contradiction. So Π is irreducible.

We now show the converse. First, note that if π is reducible then there exists a projection p on $\mathcal{H}$ which is neither 0 or 1 such that p commutes with the range of π. It is then immediate that $1\otimes p$ commutes with the range of Π and thus Π is reducible.

Assume now that there exists $g\in G\setminus\{e\}$ such that π and $\pi\circ\alpha_g$ are unitarily equivalent. Then there exists a unitary V such that for every $a\in A$:

$$\pi(a) = V\pi\left(\alpha_g(a)\right)V^*.$$

Let us show that $\rho(g)\otimes V$ is in the commutant of $\Pi\left(A\rtimes_\alpha G\right)$. We only need to check that it commutes with $\Pi(a)$ for $a\in A$.

$$\begin{aligned}(\rho(g)\otimes V)\,\Pi(a)(\delta_h\otimes\xi) &= \delta_{hg}\otimes V\pi\left(\alpha_{h^{-1}}(a)\right)\xi, \text{ and}\\ \Pi(a)(\rho(g)\otimes V)(\delta_h\otimes\xi) &= \delta_{hg}\otimes\pi\left(\alpha_{g^{-1}}\alpha_{h^{-1}}(a)\right)V\xi.\end{aligned}$$

Since $V\pi\left(\alpha_{h^{-1}}(a)\right) = \pi\left(\alpha_{g^{-1}}\alpha_{h^{-1}}(a)\right)V$, we conclude that the two quantities are equal, and that Π is reducible.

Hence, if Π is irreducible, then π is irreducible and not equivalent to $\pi\circ\alpha_g$ for any $g\in G\setminus\{e\}$. $\square$

Theorem (2.4) provides us with a possible family of irreducible representations of the crossed-product. The representations given in Theorem (2.4) are called *regular representations* of $A\rtimes_\alpha G$, whether or not they are irreducible.

However, we shall see that there are many irreducible representations of $A\rtimes_\alpha G$ which are not regular. Easy examples are provided by actions of finite cyclic groups by inner automorphisms on full matrix algebras, where the identity representation is in fact the only irreducible

representation of the crossed-product. More generally, the conditions that π is irreducible and $\pi \circ \alpha_g$ are not equivalent for $g \in G \backslash \{e\}$ are not necessary. These observations will be placed into a more general context as we now address the question raised at the start of this paper in the next section.

3. Actions of Finite Groups

This section is concerned with establishing results describing the irreducible representations of crossed-products by finite groups. The main tool for our study is to understand such actions from the perspective of the spectrum of the C*-algebra. In this paper, the spectrum $\widehat{A}$ of a C*-algebra A is the set of unitary equivalence classes of irreducible representations of A.

We start by two simple observations. Let α be the action of a finite group G on some unital C*-algebra A. Let π_1 and π_2 be two equivalent irreducible representations of A, so that there exists a unitary u such that $u\pi_1 u^* = \pi_2$. Then trivially $u\left(\pi_1 \circ \alpha_{g^{-1}}\right) u^* = \pi_2 \circ \alpha_{g^{-1}}$ for all $g \in G$. Moreover, $\pi_1 \circ \alpha_{g^{-1}}$ has the same range as π_1 and thus is irreducible as well. These two remarks show that for all $g \in G$ there exists a map $\widehat{\alpha_g}$ of G on $\widehat{A}$ defined by mapping the class of an irreducible representation π of A to the class of $\pi \circ \alpha_{g^{-1}}$. Since $(\pi \circ \alpha_{g^{-1}}) \circ \alpha_{h^{-1}} = \pi \circ \alpha_{(hg)^{-1}}$, we have $\widehat{\alpha}_h \circ \widehat{\alpha}_g = \widehat{\alpha}_{hg}$, and trivially $\widehat{\alpha}_e$ is the identity on $\widehat{A}$. Thus $\widehat{\alpha}$ is an action of G on $\widehat{A}$.

Given a representation Π of the crossed-product $A \rtimes_\alpha G$, we define the support of Π as the subset Σ of $\widehat{A}$ of all classes of irreducible representations of A weakly contained in $\Pi_{|A}$. Our main interest are in the support of irreducible representations of $A \rtimes_\alpha G$ which we now prove are always finite.

3.1. **Finiteness of irreducible supports.** Let G be a finite group of neutral element e. Let $\widehat{G}$ be the dual of G i.e. the set of unitary equivalence classes of irreducible representations of G. By [6, 15.4.1, p. 291], the cardinal of $\widehat{G}$ is given by the number of conjugacy classes of G, so $\widehat{G}$ is a finite set. Let $\rho \in \widehat{G}$ and λ be any irreducible representation of G of class ρ acting on a Hilbert space $\mathcal{H}$. Then $\overline{\lambda}$ is the (irreducible) representation $g \in G \mapsto \lambda(g)$ acting on the conjugate Hilbert space $\overline{\mathcal{H}}$ [6, 13.1.5, p. 250]. We define $\overline{\rho}$ as the class of representations unitarily equivalent to $\overline{\lambda}$.

Let B be a unital C*-algebra and α an action of G on B by *-automorphisms. We now recall from [7] the definition and elementary

properties of the spectral subspaces of B for the action α of G. Let $\rho \in \widehat{G}$. The character of ρ is denoted by χ_ρ. All irreducible representations of G whose class in $\widehat{G}$ is ρ act on vector spaces of the same dimension which we denote by $\dim \rho$. We recall from [6, 15.3.3, p. 287] that for any $\rho, \rho' \in \widehat{G}$ we have:

$$\chi_\rho(e) = \dim \rho$$

and:

$$\chi_\rho * \chi_{\rho'}(g) = \sum_{h \in G} \chi_\rho(h)\chi_{\rho'}(gh^{-1}) = \begin{cases} 0 & \text{if } \rho \neq \rho', \\ (\dim \rho)^{-1} \chi_\rho(g) & \text{if } \rho = \rho'. \end{cases}$$

The spectral subspace of B for α associated to $\rho \in \widehat{G}$ is the space B_ρ defined by:

$$B_\rho = \left\{ \frac{\dim(\rho)}{|G|} \sum_{g \in G} \chi_{\overline{\rho}}(g)\alpha_g(b) : b \in B \right\},$$

i.e. the range of the Banach space operator on B defined by:

$$P_\rho : b \in B \mapsto \frac{\dim(\rho)}{|G|} \sum_{g \in G} \chi_{\overline{\rho}}(g)\alpha_g(b). \tag{3.1}$$

In particular, the spectral subspace associated to the trivial representation is the fixed point C*-subalgebra B_1 of B for the action α of G. Now, we have:

$$\begin{aligned} P_\rho\left(P_{\rho'}(a)\right) &= \frac{\dim(\rho)}{|G|}\frac{\dim(\rho')}{|G|} \sum_{g \in G}\sum_{h \in G} \chi_{\overline{\rho}}(g)\chi_{\overline{\rho'}}(h)\alpha_{gh}(a) \\ &= \frac{\dim(\rho)}{|G|}\frac{\dim(\rho')}{|G|} \sum_{g \in G}\left(\sum_{h \in G} \chi_{\overline{\rho}}(gh^{-1})\chi_{\overline{\rho'}}(h)\right)\alpha_g(a) \\ &= \begin{cases} 0 & \text{if } \rho \neq \rho' \\ \frac{\dim(\rho)}{|G|} \sum_{g \in G} \chi_{\overline{\rho}}(g)\alpha_g(a) & \text{if } \rho = \rho'. \end{cases} \qquad (3.2) \end{aligned}$$

Hence $P_\rho^2 = P_\rho$ so P_ρ is a Banach space projection and $P_\rho P_{\rho'} = 0$ for all $\rho' \neq \rho$ so these projections are pairwise orthogonal.

Moreover, for any $g, h \in G$, from [6, 15.4.2 (2) p. 292]:

$$\sum_{\rho \in \widehat{G}} \chi_\rho(g)\overline{\chi_\rho(h)} = \begin{cases} \frac{|G|}{C(g)} & \text{if } g \text{ is conjugated with } h, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

where for $g \in G$ the quantity $C(g)$ is the number of elements in G conjugated to g. In particular, note that since $g \in G \setminus \{e\}$ is not

conjugated to e, we have by Equality (3.3) that:

$$\sum_{\rho\in\widehat{G}} \chi_\rho(g)\dim\rho = \sum_{\rho\in\widehat{G}} \chi_\rho(g)\overline{\chi_\rho(e)} = 0. \tag{3.4}$$

Furthermore, because each irreducible representation ρ of G appears with multiplicity $\dim\rho$ in the left regular representation of G one can show [6, 15.4.1, p. 291] that:

$$\sum_{\rho\in\widehat{G}} (\dim\rho)^2 = |G|\,. \tag{3.5}$$

Hence for all $b\in B$:

$$\begin{aligned}
\sum_{\rho\in\widehat{G}} P_\rho(b) &= \sum_{\rho\in\widehat{G}} \frac{\dim(\rho)}{|G|}\sum_{g\in G}\chi_\rho(g)\alpha_g(b)\\
&= \frac{1}{|G|}\sum_{g\in G}\left(\sum_{\rho\in\widehat{G}}\dim(\rho)\,\chi_\rho(g)\right)\alpha_g(b)\\
&= \frac{1}{|G|}\left(\sum_{\rho\in\widehat{G}}\dim(\rho)\,\chi(e)\right)\alpha_e(b) \text{ by Equality (3.4)}\\
&= \left(\frac{1}{|G|}\sum_{\rho\in\widehat{G}}\dim(\rho)^2\right) b = b \text{ by Equality (3.5).} \qquad (3.6)
\end{aligned}$$

Hence $\sum_{\rho\in\widehat{G}} P_\rho = \mathrm{Id}_B$. Thus by (3.2) and (3.6) we have:

$$B = \bigoplus_{\rho\in\widehat{G}} B_\rho. \tag{3.7}$$

We now establish that the restriction of any irreducible representation of a crossed-product of some unital C*-algebra A by G is the direct sum of finitely many irreducible representations of A.

Theorem 3.1. *Let G be a finite group and A a unital C*-algebra. Let α be an action of G by *-automorphism on A. Let Π be an irreducible representation of $A\rtimes_\alpha G$ on some Hilbert space $\mathcal{H}$. We denote by U^g the canonical unitary in $A\rtimes_\alpha G$ corresponding to $g\in G$. Then:*

- *The action $g\mapsto \mathrm{Ad}\,\Pi(U^g)$ on $\mathcal{B}(\mathcal{H})$ leaves the commutant $\Pi(A)'$ of $\Pi(A)$ invariant, and thus defines an action β of G on $\Pi(A)'$,*
- *The action β is ergodic on $\Pi(A)'$,*
- *The Von Neumann algebra $\Pi(A)'$ is finite dimensional,*

- *The representation $\Pi_{|A}$ of A is equivalent to the direct sum of finitely many irreducible representations of A.*

Proof. Let $\mathfrak{M} = \Pi(A)'$. Denote $U_\Pi^g = \Pi(U^g)$ for all $g \in G$. Let $T \in \mathfrak{M}$. Let $a \in A$ and $g \in G$. Then:

$$\begin{aligned} U_\Pi^g T U_\Pi^{g*} \Pi(a) &= U_\Pi^g T U_\Pi^{g*} \Pi(a) U_\Pi^g U_\Pi^{g*} \\ &= U_\Pi^g T \Pi\left(\alpha_{g^{-1}}(a)\right) U_\Pi^{g*} \\ &= U_\Pi^g \Pi\left(\alpha_{g^{-1}}(a)\right) T U_\Pi^{g*} \\ &= U_\Pi^g U_\Pi^{g*} \Pi(a) U_\Pi^g T U_\Pi^{g*} \\ &= \Pi(a) U_\Pi^g T U_\Pi^{g*}. \end{aligned}$$

Hence $U_\Pi^g T U_\Pi^{g*} \in \mathfrak{M}$ for all $g \in G$ and $T \in \mathfrak{M}$. Define $\beta_g(T) = U_\Pi^g T U_\Pi^{g*}$ for all $g \in G$ and $T \in \mathfrak{M}$. Then $g \in G \mapsto \beta_g$ is an action of G on $\mathfrak{M}$.

Let now $T \in \mathfrak{M}$ such that $\beta_g(T) = T$ for all $g \in G$. Then T commutes with U_Π^g for all $g \in G$. Moreover by definition of $\mathfrak{M}$, the operator T commutes with $\Pi(A)$. Hence T commutes with Π which is irreducible, so T is scalar. Hence β is ergodic.

Let ρ be an irreducible representation of G (since G is finite, ρ is finite dimensional). By [7, Proposition 2.1], the spectral subspace $\mathfrak{M}_\rho$ of $\mathfrak{M}$ for β associated to ρ is finite dimensional. Since $\mathfrak{M} = \oplus_{\rho \in \widehat{G}} \mathfrak{M}_\rho$ by Equality (3.7) and since $\widehat{G}$ is finite by [6, 15.4.1, p. 291] we conclude that $\mathfrak{M}$ is finite dimensional.

Denote $\Pi_{|A}$ by π_A. Let $p_1, \ldots, p_k$ be projections in $\mathfrak{M}$, all minimal and such that $\sum_{i=1}^k p_i = 1$. Let $i \in \{1, \ldots, k\}$. Then by definition of $\mathfrak{M}$, the projection p_i commutes with π_A. Hence $p_i \pi_A p_i$ is a representation of A. Let q be a projection of $p_i \mathcal{H}$ such that p_i commutes with $p_i \pi_A p_i$. Then $q \leq p_i$ and $q \in \mathfrak{M}$, so $q \in \{0, p_i\}$ since p_i is minimal. Hence $p_i \pi_A p_i$ is an irreducible representation of A. Therefore:

$$\begin{aligned} \pi_A &= \left(\sum_{i=1}^k p_i\right) \pi_A \text{ since } \sum_{i=1}^k p_i = 1, \\ &= \sum_{i=1}^k p_i \pi_A p_i \text{ since } p_i = p_i^2 \in \mathfrak{M}. \end{aligned}$$

Hence π_A is the direct sum of finitely many irreducible representations of A. $\square$

3.2. **Minimality of the irreducible supports.** The following is our key observation which will drive the proofs in this section:

Observation 3.2. *Let Π be an irreducible representation of $A \rtimes_\alpha G$ and let $\pi_A = \Pi_{|A}$. Then for each $g \in G$ the representations π_A and $\pi_A \circ \alpha_g$ are unitarily equivalent. Hence, the decompositions in direct sums of irreducible representations of A for π_A and $\pi_A \circ \alpha_g$ are the same.*

This observation is the basis of the next lemma, which is instrumental in the proof of the theorem to follow.

Lemma 3.3. *Let α be an action of a finite group G on a unital C^*-algebra A. Let Π be an irreducible representation of $A \rtimes_\alpha G$ and let π_A be the restriction of Π to A. Then there exists a finite subset Σ of the spectrum $\widehat{A}$ of A such that all irreducible subrepresentations of π_A are in Σ. Moreover, all the elements of Σ in a given orbit for $\widehat{\alpha}$ have the same multiplicity in π_A.*

Proof. Let Σ be the subset of the spectrum $\widehat{A}$ of A consisting of all classes of irreducible representations weakly contained in π_A. By Theorem (3.1), since Π is irreducible, π_A is a finite direct sum of irreducible representations of A so Σ is nonempty and finite.

Let $g \in G$. Now, by Observation (3.2), since $\pi_A \circ \alpha_{g^{-1}}$ is unitarily equivalent to π_A, its decomposition in irreducible representations is the same as the one for π_A. Thus, if $\eta \in \Sigma$ then $\widehat{\alpha}_g(\eta) \in \Sigma$. Since $\widehat{\alpha}_g$ is a bijection on $\widehat{A}$ and thus is injective, and since Σ is finite, $\widehat{\alpha}_g$ is a permutation of Σ.

Let Σ_α be the orbit of $\varphi \in \Sigma$ under $\widehat{\alpha}$ and write $\pi_A = \pi_1 \oplus \ldots \oplus \pi_k$ using Theorem (3.1), where $\pi_1, \ldots, \pi_k$ are irreducible representations of A, with the class of π_1 being φ. Now, for $g \in G$, let $n_{1,g}, \ldots, n_{m(g),g}$ be the integers between 1 and k such that $\pi_{n_{i,g}}$ is equivalent to $\pi_1 \circ \alpha_g$. In particular, $m(g)$ is the multiplicity of $\pi_1 \circ \alpha_g$ in π_A. Then $\left(\pi_{n_{1,e}} \oplus \ldots \oplus \pi_{n_{m(1),e}}\right) \circ \alpha_g$ must be the subrepresentation $\pi_{n_{1,g}} \oplus \ldots \oplus \pi_{n_{m(g),g}}$ of π_A. So $m(g) = m(e)$ by uniqueness of the decomposition. Hence for all g the multiplicity of $\widehat{\alpha}_g(\varphi)$ is the same as the multiplicity of φ. $\square$

We now establish the main theorem of this paper, describing the structure of irreducible representations of crossed-products by finite groups. A *unitary projective representation* of G is a map Λ from G into the group of unitaries on some Hilbert space such that there exists a complex valued 2-cocycle σ on G satisfying for all $g, h \in G$ the identity $\Lambda_{gh} = \sigma(g,h)\Lambda_g\Lambda_h$.

Theorem 3.4. *Let G be a finite group and α be an action of G on a unital C^*-algebra A by *-automorphisms. Let Π be an irreducible representation of $A \rtimes_\alpha G$ on some Hilbert space $\mathcal{H}$. Then there exists a subgroup H of G and a representation π of A on some Hilbert space $\mathcal{J}$ such that, up to conjugating Π by some fixed unitary, and denoting the index of H in G by $m = G : H$ we have the following:*

For any subset $\{g_1, \ldots, g_m\}$ of G such that g_1 is the neutral element of G and $Hg_j \cap Hg_i = \{g_1\}$ for $i \neq j$ while $G = \cup_{j=1}^m Hg_j$, we have:

(1) *The representations $\pi \circ \alpha_{g_i}$ and $\pi \circ \alpha_{g_j}$ are disjoint for $i, j \in \{1, \ldots, m\}$ and $i \neq j$ (so in particular, they are not unitarily equivalent),*
(2) *There exists an irreducible representation π_1 of A on a Hilbert subspace $\mathcal{H}_1$ of $\mathcal{J}$ and some integer r such that $\mathcal{J} = \mathbb{C}^r \otimes \mathcal{H}_1$ and $\pi = 1_{\mathbb{C}^r} \otimes \pi_1$,*
(3) *For any $h \in H$ there exists a unitary V^h on $\mathcal{H}_1$ such that $V^h \pi_1 \left(V^h\right)^* = \pi_1 \circ \alpha_h$, and $h \in H \mapsto V^h$ is a unitary projective representation of H on $\mathcal{H}_1$,*
(4) *We have $\mathcal{H} = \mathcal{J}_{g_1} \oplus \ldots \oplus \mathcal{J}_{g_m}$ where for all $i = 1, \ldots, m$ the space $\mathcal{J}_{g_i}$ is an isometric copy of $\mathcal{J}$,*
(5) *In this decomposition of $\mathcal{H}$ we have for all $a \in A$ that:*

$$\Pi(a) = \begin{bmatrix} \pi(a) & & & \\ & \pi \circ \alpha_{g_2}(a) & & \\ & & \ddots & \\ & & & \pi \circ \alpha_{g_m}(a) \end{bmatrix} \tag{3.8}$$

(6) *In this same decomposition, for every g there exists a permutation σ^g of $\{1, \ldots, m\}$ and unitaries $U_i^g : \mathcal{J}_{g_i} \longrightarrow \mathcal{J}_{\sigma^g(g_i)}$ such that:*

$$\Pi(U^g) = \left[U_j^g \delta_i^{\sigma^g(j)}\right]_{i,j=1,\ldots,m}$$

where δ is the Kronecker symbol:

$$\delta_a^b = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Moreover:

$$H = \{g \in G : \sigma^g(1) = 1\}.$$

(7) *The representation Ψ of $A \rtimes_\alpha H$ on $\mathcal{J}$ defined by $\Psi(a) = \pi(a)$ for all $a \in A$ and $\Psi(U^h) = U_1^h$ for $h \in H$ is irreducible. Moreover, there exists an irreducible unitary projective representation Λ of G on $\mathbb{C}^r$ such that on $\mathcal{J} = \mathbb{C}^r \otimes \mathcal{H}_1$, while $\Psi(a) = 1_{\mathbb{C}^r} \otimes \pi_1(a)$, we also have $\Psi(U^h) = U_1^h = \Lambda_h \otimes V^h$.*

Proof. Let Π be an irreducible representation of $A \rtimes_\alpha G$. Denote $\Pi_{|A}$ by π_A. By Theorem (3.1), there exists a nonzero natural integer k and irreducible representations $\pi_1, \ldots, \pi_k$ of A, acting respectively on Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_k$ such that up to a unitary conjugation of Π, we have $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_k$ and in this decomposition, for all $a \in A$:

$$\pi_A(a) = \begin{bmatrix} \pi_1(a) & & & \\ & \pi_2(a) & & \\ & & \ddots & \\ & & & \pi_k(a) \end{bmatrix}.$$

At this stage, the indexing of the irreducible subrepresentations of π_A is only defined up to a permutation of $\{1, \ldots, k\}$. We start our proof by making a careful choice of such an indexing. To do so, first choose π_1 arbitrarily among all irreducible subrepresentations of π_A. Our next step is to set:

$$H = \{g \in G : \pi_1 \circ \alpha_g \text{ is equivalent to } \pi_1\}.$$

We now show that H is a subgroup of G. For all $h \in H$ we denote by V^h the (unique, up to a scalar multiple) unitary such that $V^h \pi_1 \left(V^h\right)^* = \pi_1 \circ \alpha_h$. Then if $g, h \in H$ we have:

$$\begin{aligned} \pi_1 \circ \alpha_{gh^{-1}} &= (\pi_1 \circ \alpha_g) \circ \alpha_{h^{-1}} = V^g \left(\pi_1 \circ \alpha_{h^{-1}}\right) V^{g^{-1}} \\ &= V^g V^{h^{-1}} \pi_1 V^h V^{g^{-1}} \end{aligned}$$

so $\pi_1 \circ \alpha_{gh^{-1}}$ is unitarily equivalent to π_1 and thus $gh^{-1} \in H$ by definition. Since H trivially contains the neutral element of G, we conclude that H is a subgroup of G.

Let $\{g_1, \ldots, g_m\}$ a family of right coset representatives such that g_1 is the neutral element of G [10, p. 10], i.e. such that for $i \neq j$ we have $Hg_j \cap Hg_i = \{g_1\}$ while $G = \cup_{j=1}^m Hg_j$. In particular, for $i \in \{2, \ldots, m\}$ we have $g_1 \neq g_i$ and by definition of H this implies that $\pi_1 \circ \alpha_{g_i}$ is not equivalent to π_1.

Then let $\pi_2, \ldots, \pi_{n_1}$ be all the representations equivalent to π_1. We then choose π_{n_1+1} to be a subrepresentation of π_A equivalent to $\pi_1 \circ \alpha_{g_1}$. Again, we let $\pi_{n_1+1}, \ldots, \pi_{n_2}$ be all the representations which are equivalent to π_{n_1+1}. More generally, we let $\pi_{n_j+1}, \ldots, \pi_{n_{j+1}}$ be all the subrepresentations of π_A equivalent to $\pi_1 \circ \alpha_{g_j}$ for all $j \in \{1, \ldots, m\}$. All other irreducible subrepresentations of π_A left, if any, are indexed from $n_m + 1$ to k and we denote their direct sum by Λ.

Note that Λ contains no subrepresentation equivalent to any representation $\pi_1 \circ \alpha_g$ for any $g \in G$. Indeed, if $g \in G$ then there exists

$h \in H$ and a unique $j \in \{1, \ldots, m\}$ such that $g = hg_j$. Thus:

$$\pi_1 \circ \alpha_g = \pi_1 \circ \alpha_h \circ \alpha_{g_j} = V^h \left(\pi_1 \circ \alpha_{g_j}\right) V^{-h}$$

and thus $\pi_1 \circ \alpha_g$ is equivalent to one of the representations $\pi_1, \ldots, \pi_{n_m}$ by construction. Also note that if $\pi_1 \circ \alpha_{g_i}$ is equivalent to $\pi_1 \circ \alpha_{g_j}$ then $g_i g_j^{-1} \in H$ which contradicts our choice of $\{g_1, \ldots, g_m\}$ unless $i = j$. Hence, for $i \neq j$ the representations $\pi_1 \circ \alpha_{g_j}$ and $\pi_1 \circ \alpha_{g_i}$ are not equivalent.

Now, if $\varphi_1, \ldots, \varphi_m$ represent the unitary-equivalence classes of the representations $\pi_1, \pi_1 \circ \alpha_{g_1}, \ldots, \pi_1 \circ \alpha_{g_m}$ then $\Sigma_1 = \{\varphi_1, \ldots, \varphi_m\}$ is the orbit of φ_1 for the action $\widehat{\alpha}$ of G on $\widehat{A}$. Therefore, there exists $r \geq 1$ such that $n_j = jr + 1$ for all $j = 1, \ldots, m$ by Lemma (3.3), i.e. all the representations $\pi_1 \circ \alpha_{g_i}$ $(i = 1, \ldots, m)$ have multiplicity r in π_A.

Thus, (up to equivalence on Π) and writing $\mathcal{H} = \oplus_{i=1}^{k} \mathcal{H}_i$ and in this decomposition:

$$\begin{aligned}
\pi_A &= \underbrace{\pi_1 \oplus \ldots \oplus \pi_r}_{\text{each equivalent to } \pi_1} \oplus \underbrace{\pi_{r+1} \oplus \ldots \oplus \pi_{2r}}_{\text{each equivalent to } \pi_1 \circ \alpha_{g_1}} \oplus \cdots \\
&\qquad \ldots \oplus \pi_{mr} \oplus \underbrace{\pi_{mr+1} \oplus \ldots \oplus \pi_k}_{\Lambda} \\
&= \underbrace{\pi_1 \oplus \ldots \oplus \pi_{n_m}}_{\text{disjoint from } \Lambda.} \oplus \Lambda.
\end{aligned} \tag{3.10}$$

Let $g \in G$. Still in the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_k$ with our choice of indexing, let us write:

$$\Pi\left(U^g\right) = U^g_\Pi = \left[\begin{array}{cccc} a^g_{11} & a^g_{12} & \cdots & a^g_{1k} \\ a^g_{21} & a^g_{22} & \cdots & a^g_{2k} \\ \vdots & & \ddots & \vdots \\ a^g_{k1} & a^g_{k2} & \cdots & a^g_{kk} \end{array}\right]$$

for some operators a^g_{ij} from $\mathcal{H}_j$ to $\mathcal{H}_i$ with $i, j = 1, \ldots, k$.

Since $U^g_\Pi \pi_A(a) = \pi_A(\alpha_g(a)) U^g_\Pi$, we can write:

$$\begin{aligned}
&\left[\begin{array}{cccc} a^g_{11}\pi_1 & a^g_{12}\pi_2 & \cdots & a^g_{1k}\pi_k \\ a^g_{21}\pi_1 & a^g_{22}\pi_2 & \cdots & a^g_{2k}\pi_k \\ \vdots & & \ddots & \vdots \\ a^g_{k1}\pi_1 & a^g_{k2}\pi_2 & \cdots & a^g_{kk}\pi_k \end{array}\right] \\
= &\left[\begin{array}{cccc} (\pi_1 \circ \alpha_g)\, a^g_{11} & (\pi_1 \circ \alpha_g)\, a^g_{12} & \cdots & (\pi_1 \circ \alpha_g)\, a^g_{1k} \\ (\pi_2 \circ \alpha_g)\, a^g_{21} & (\pi_2 \circ \alpha_g)\, a^g_{22} & \cdots & (\pi_2 \circ \alpha_g)\, a^g_{2k} \\ \vdots & & \ddots & \vdots \\ (\pi_k \circ \alpha_g)\, a^g_{k1} & (\pi_k \circ \alpha_g)\, a^g_{k2} & \cdots & (\pi_k \circ \alpha_g)\, a^g_{kk} \end{array}\right].
\end{aligned} \tag{3.11}$$

As a consequence of Equality (3.11), we observe that for all $i,j \in \{1,\ldots,k\}$ we have:

$$a_{ij}^{g}\pi_j = (\pi_i \circ \alpha_g)\, a_{ij}^{g}. \tag{3.12}$$

First, let $i > mr$. Then the equivalence class of π_i is not in the orbit Σ_1 of φ_1 for $\widehat{\alpha}$ by construction. Hence $\pi_i \circ \alpha_g$ is not unitarily equivalent to $\pi_1 \circ \alpha_\gamma$ for any $\gamma \in G$. On the other hand, let $j \leq mr$. The representation π_j is equivalent to $\pi_1 \circ \alpha_{g_l}$ for some $l \in \{1,\ldots,m\}$ by our choice of indexing. Therefore, $\pi_i \circ \alpha_g$ and π_j are not unitarily equivalent, yet they both are irreducible representations of A. Hence by Lemma (2.3) applied to Equality (3.12) we conclude that $a_{ij}^{g} = 0$. Similarly, π_i and $\pi_j \circ \alpha_g$ are not equivalent so $a_{ji}^{g} = 0$ as well.

Hence:

$$U_{\Pi}^{g} = \left[\begin{array}{cccccc} a_{11}^{g} & \cdots & a_{1mr}^{g} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{mr1}^{g} & \cdots & a_{mr,mr}^{g} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_{mr+1,mr+1}^{g} & \cdots & a_{mr+1,k}^{g} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{k,mr+1}^{g} & \cdots & a_{kk}^{g} \end{array}\right].$$

If we assume that $n_m = mr < k$ then for all $g \in G$ the unitary U_{Π}^{g} commutes with the nontrivial projection $\underbrace{0 \oplus \ldots \oplus 0}_{mr \text{ times}} \oplus \underbrace{1 \oplus \ldots \oplus 1}_{k-mr \text{ times}}$ of $\mathcal{H}$, and so does π_A. Yet Π is irreducible, so this is not possible and thus $n_m = k$. Thus $\Sigma = \Sigma_1$ is an orbit of a single $\varphi \in \widehat{A}$ for $\widehat{\alpha}$ and there is no Λ left in Equality (3.10). In particular, the cardinal of Σ_1 is m.

Since by construction π_{jr+z} is unitarily equivalent to $\pi_1 \circ \alpha_{g_z}$ for all $j = 0,\ldots,m-1$ and $z = 1,\ldots,r$, there exists a unitary ω_{jr+z} from $\mathcal{H}_1$ onto $\mathcal{H}_{jr+z}$ such that $\omega_{jr+z}(\pi_1 \circ \alpha_{g_z})\,\omega_{jr+z}^{*} = \pi_{jr+z}$ (note that we can choose $\omega_1 = 1$). We define on $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_k$ the diagonal unitary:

$$\Omega = \left[\begin{array}{ccc} \omega_1^{*} & & \\ & \ddots & \\ & & \omega_k^{*} \end{array}\right].$$

Denote by $\operatorname{Ad}\Omega$ is the *-automorphism on the C*-algebra of bounded operators on $\mathcal{H}$ defined by $T \mapsto \Omega T \Omega^{*}$. Then up to replacing Π by $\operatorname{Ad}\Omega \circ \Pi$, we can assume that $\pi_{jr+z} = \pi_1 \circ \alpha_{g_z}$ for all $j \in \{1,\ldots,m\}$ and $z \in \{1,\ldots,r\}$. Given an irreducible representation η of A and any nonzero natural integer z we shall denote by $z \cdot \eta$ the representation $\underbrace{\eta \oplus \ldots \oplus \eta}_{z \text{ times}}$. Thus, if we set $\pi = r \cdot \pi_1$ we see that π_A can be written as

in Equality (3.8) with $\pi \circ \alpha_{g_i}$ disjoint from $\pi \circ \alpha_{g_j}$ for $i, j \in \{1, \ldots, m\}$ and $i \neq j$.

Let again $g \in G$. We now use the same type of argument to show that U_Π^g is a "unitary-permutation shift". Let $j \in \{0, \ldots, m-1\}$. Let $q \in \{1, \ldots, m\}$ such that $g_j g \in H g_q$ — by our choice of $g_1, \ldots, g_m$ there is a unique such q. Let $i \in \{0, \ldots, m-1\} \setminus \{q\}$ and $z, h \in \{1, \ldots, r\}$. By construction, the representation $(r \cdot \pi_{rj+h}) \circ \alpha_g$ is unitarily equivalent to $r \cdot \pi_{rq+h}$ and disjoint from $r \cdot \pi_{ri+z}$. Yet by Equality (3.12) we have again that:

$$a^g_{ri+z,rj+h} \pi_{ri+z} = (\pi_{rj+h} \circ \alpha_g)\, a^g_{ri+z,rj+h}.$$

Thus $a^g_{ri+z,rj+h} = 0$ by Lemma (2.3) since π_{ri+z} and $\pi_{rj+h} \circ \alpha_g$ are not equivalent yet irreducible. Thus, if for all $z \in \{0, \ldots, m-1\}$ we define the Hilbert subspace $\mathcal{J}_z = \mathcal{H}_{zr+1} \oplus \ldots \oplus \mathcal{H}_{(z+1)r}$ of $\mathcal{H}$ then we conclude that $U_\Pi^g(\mathcal{J}_j) \subseteq \mathcal{J}_q$ and $\mathcal{H} = \mathcal{J}_0 \oplus \ldots \oplus \mathcal{J}_{m-1}$. Moreover, by uniqueness of q we also obtain that:

$$U_\Pi^{g^{-1}}(\mathcal{J}_q) \subseteq \mathcal{J}_j \tag{3.13}$$

and thus $U_\Pi^g(\mathcal{J}_j) = \mathcal{J}_q$. Define $\sigma^g(j) = q$. Then σ^g is a surjection of the finite set $\{1, \ldots, m\}$ by (3.13), so σ^g is a permutation. If δ is defined as in Equality (3.9) then, if we set $U_j^g = U^g_{\Pi|\mathcal{J}_j}$ then:

$$(r \cdot (\pi_j \circ \alpha_g))\, U_i^g = U_i^g\, (r \cdot \pi_q) \tag{3.14}$$

and

$$U_\Pi^g = \left[U_j^g \delta_i^{\sigma^g(j)} \right]_{i,j}$$

for all $i = 1, \ldots, m$.

Since U_Π is unitary, so are the operators $U_1, \ldots, U_m$. In particular, $\mathcal{J}_j$ and $\mathcal{J}_0$ are isometric Hilbert spaces for all $j = 0, \ldots, m-1$. Note that $(r \cdot \pi_1) \circ \alpha_{g_i}$ acts on $\mathcal{J}_{i-1}$ for $i = 1, \ldots, m$ by construction. We now denote $r \cdot \pi_1$ by π and $\mathcal{J} = \mathcal{J}_0$.

Now, by construction $\sigma^g(1) = 1$ if and only if there exists an operator W on $\mathcal{J}_1 \oplus \ldots \oplus \mathcal{J}_{m-1}$ such that $U_\Pi^g = U_1^g \oplus W$, which is equivalent to $U_1^g \pi_1 = (\pi_1 \circ \alpha_g)\, U_1^g$. By construction, this is possible if and only if $g \in H$.

Let now $h \in H$. Hence $U_1^h \pi U_1^{h^{-1}} = \pi \circ \alpha_h$. If we set $\Psi(a) = \pi(a)$ and $\Psi(U^h) = U_1^h$, we thus define a representation of $A \rtimes_\alpha H$ on $\mathcal{J}_0$. Let $b \in A \rtimes_\alpha H$. Then there exists $g \in G \mapsto a_g$ such that $b = \sum_{g \in G} a_g U^g$. Hence $\Pi(b) = \sum_{g \in G} \pi_A(a_g) U_\Pi^g$. Let Q be the projection of $\mathcal{H}$ on $\mathcal{J}_0$.

Then:

$$\begin{aligned}
Q\Pi(b)Q &= \sum_{g\in G}\pi(a_g)QU_\Pi^g Q \\
&= \sum_{h\in H}\pi(a_h)U_1^h = \Psi\left(\sum_{h\in H}a_h U^h\right). \qquad (3.15)
\end{aligned}$$

Since Π is irreducible, the range of Π is WOT dense by the double commutant theorem. Hence, since the multiplication on the left and right by a fixed operator is WOT continuous, we conclude that $Q\Pi Q$ is WOT dense in $\mathcal{B}(Q\mathcal{H})$. Therefore, by Equality (3.15), we conclude by the double commutant Theorem again that Ψ is an irreducible representation of $A \rtimes_\alpha H$.

Last, note that since π_1 is irreducible, if $h,g\in H$ then since:

$$V^{h^{-1}}V^{g^{-1}}V^{gh}\pi_1 = \pi_1 V^{h^{-1}}V^{g^{-1}}V^{gh}$$

there exists $\lambda_{g,h}\in\mathbb{T}$ such that $V^{gh}=\lambda_{g,h}V^gV^h$. Hence $g\in H\mapsto V^g$ is a projective representation of H on $\mathcal{H}_1$. Note that although the unitaries V^h are only defined up to a scalar, there is no apparent reason why one could choose λ to be the trivial cocycle unless the second cohomology group of H is trivial. We now note that $\mathcal{J}_0=\mathcal{J}=\mathbb{C}^r\otimes\mathcal{H}_1$ by construction. Now, for all $h\in H$ we set $\upsilon_h = 1_{\mathbb{C}^r}\otimes V^h$. Again, υ_h is a projective representation of H. Moreover, for $h\in H$:

$$U_1^h\upsilon_h^*\pi = \pi U_1^h\upsilon_h^*.$$

Since $\pi = r\cdot\pi_1$, Lemma (2.3) implies that there exist a unitary $\Lambda_h\in M_r(\mathbb{C})$ such that $U_1^h\upsilon_h^* = \Lambda_h\otimes 1_{\mathcal{H}_1}$. Hence $U_1^h=\Lambda_h\otimes V^h$. Now, for $h,g\in H$ we have $U_1^hU_1^g = U_1^{hg}$ which implies that:

$$\left(\Lambda_h\otimes V^h\right)\left(\Lambda_g\otimes V^g\right) = \Lambda_h\Lambda_g\otimes V^hV^g = \Lambda_{hg}\otimes V^{hg}.$$

Hence $h\mapsto\Lambda_h$ is a unitary projective representation of H on $\mathbb{C}^r$ with cocycle $\overline{\lambda}$. Moreover, if T commutes with the range of Λ then $T\otimes 1$ commutes with the range of Ψ, which contradicts the irreducibility of Ψ. Hence Λ is irreducible. This completes the description of the representation Ψ. $\square$

For generic groups, the representation Ψ of Theorem (3.4) may not be minimal, i.e. its restriction to A may be reducible. The simplest way to see this is by consider a finite group G admitting a representation Λ on $\mathbb{C}^n$ for some $n\in\mathbb{N}$. Then Λ extends to an irreducible representation Π of the crossed-product $\mathbb{C}\rtimes_\alpha G$ where α is the trivial action. Thus, $\Pi_{|\mathbb{C}}$, which decomposes into a direct sum of irreducible representations of $\mathbb{C}$, must in fact be the direct sum of n copies of the (unique) identity

representation of $\mathbb{C}$. Note that in this case $\Pi = \Psi$ using the notations of Theorem (3.4). Thus, for any $n \in \mathbb{N}$ one can find an example where Ψ is irreducible yet not minimal. This situation will be illustrated with a much less trivial example in Example (5.6) where G will be permutation group on three elements. However, the representation Ψ must be minimal when the group G is chosen to be a finite cyclic group. We develop the theory for these groups in the next section.

Because the representation Ψ of Theorem (3.4) is of central interest in the decomposition of Π, we establish the following criterion for irreducibility for such representations. Note that the next theorem also describes the situation where the commutant of Π is a factor.

Theorem 3.5. *Let H be a discrete group. Let Ψ be a representation of $A \rtimes_\alpha H$ on a Hilbert space $\mathcal{H}$ and assume there exists an irreducible representation π_1 of A on a Hilbert space $\mathcal{H}_1$ such that $\mathcal{H} = \mathbb{C}^r \otimes \mathcal{H}_1$, $\pi_1 \circ \alpha_h$ is equivalent to π_1 for all $h \in H$ and $\Psi(a) = 1_{\mathbb{C}^r} \otimes \pi(a)$ for all $a \in A$. Then there exist two unitary projective representations Λ and V of H on $\mathbb{C}^r$ and $\mathcal{H}_1$ respectively such that $\Psi(U^h) = \Lambda_h \otimes V^h$. Moreover, the following are equivalent:*

1. *Ψ is irreducible,*
2. *The representation Λ is irreducible.*

Proof. By assumption, for $h \in H$ there exists a unitary V^h such that $V^h \pi_1 \left(V^h\right)^* = \pi_1 \circ \alpha_h$ and this unitary is unique up to a constant by Lemma (2.3). From the last section of the proof of Theorem (3.4), we get that $h \in H \mapsto V^h$ is a projective representation of H for some 2-cocycle λ and, since π_1 is irreducible, there exists a projective representation Λ of H on $\mathbb{C}^r$ such that $\Psi(U^h) = \Lambda_h \otimes V^h$, and moreover if Ψ is irreducible then so is Λ.

Suppose now Λ is irreducible. Let $T \in \left[\Psi\left(A \rtimes_\alpha H\right)\right]'$. Since T commutes with $\Psi(A) = 1_{\mathbb{C}^r} \otimes \pi_1(A)$, it follows that $T = D \otimes 1_{\mathcal{H}_1}$ for some $D \in M_r(\mathbb{C})$. Now T commutes with $\Psi(U^h)$ for all $h \in H$, so D commutes with Λ_g for all $g \in H$. Hence D is scalar and Ψ is irreducible. $\square$

We also note that the group H is not a priori a normal subgroup of G. It is easy to check that the following two assertions are equivalent:

1. H is a normal subgroup of G,
2. For all $g \in G$, the unitary U_Π^g is block-diagonal in the decomposition $\mathcal{H} = \mathcal{J}_0 \oplus \ldots \oplus \mathcal{J}_{m-1}$ if and only if $g \in H$.

In particular, when G is Abelian then for $g \in G$ we have $\sigma^g(1) = 1$ if and only if $\sigma^g = \mathrm{Id}$.

We conclude by observing that the representation Ψ involves projective representations of H. We now offer an example to illustrate this situation and shows that this phenomenon occurs even when G is Abelian. We shall see in the next section that finite cyclic groups have the remarkable property that such unitary projective representations do not occur.

Example 3.6. *Let p, q be two relatively prime integers. Let $\lambda = \exp\left(2i\pi\frac{p}{q}\right)$. Denote by $\mathbb{U}_q$ the group of q^{th} roots of unity in $\mathbb{C}$. Let α be the action of $\mathbb{Z}_q$ on $C(\mathbb{U}_q)$ defined by $\alpha_1(f)(z) = f(\lambda z)$. Then the crossed-product $A = C(\mathbb{U}_q) \rtimes_\alpha \mathbb{Z}_q$ is isomorphic to $M_q(\mathbb{C})$. The canonical unitary is identified under this isomorphism with:*

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

while the generator $z \in \mathbb{U}_q \mapsto z$ of $C(\mathbb{U}_q)$ is mapped to:

$$V = \begin{bmatrix} 1 & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda^{q-1} \end{bmatrix}.$$

The dual action γ of the Abelian group $G = \mathbb{Z}_q \times \mathbb{Z}_q$ on $C(\mathbb{U}_q) \rtimes_\alpha \mathbb{Z}_q$ can thus be described by:

$$\gamma_{z,z'}(U) = \exp\left(2i\pi\frac{pz}{q}\right)U \text{ and } \gamma_{z,z'}(V) = \exp\left(2i\pi\frac{pz'}{q}\right)V$$

for all $(z, z') \in G$. Now, for $(z, z') \in G$ we set $\Lambda(z, z') = \lambda^{zz'}U^zV^{z'}$. Note that G is generated by $\zeta = (1, 0)$ and $\xi = (0, 1)$ and $\Lambda(\zeta) = U$ while $\Lambda(\xi) = V$. Since $VU = \lambda UV$, the map Λ is a unitary projective representation of G on $\mathbb{C}^2$ associated to the group cohomology class of $\exp(i\pi\sigma)$ where σ is defined by:

$$\sigma((z, z'), (y, y')) = \frac{p}{q}(zy' - z'y).$$

Moreover, the dual action is of course an inner action, and more precisely:

$$\begin{aligned} \gamma_{z,z'}(a) &= U^zV^{z'}aV^{-z'}U^{-z} \\ &= \Lambda(z, z')a\Lambda(z, z')^*. \end{aligned}$$

We let $\Lambda' : z, z' \in G \mapsto \Lambda(z', z)$. *Then an easy computation shows that* Λ' *is a unitary projective representation of* G *on* $\mathbb{C}^2$ *associated to the cocycle defined by* $\exp(-i\pi\sigma)$, *and* $\Lambda'(\zeta) = V$ *and* $\Lambda'(\xi) = U$.

Let $B = A \rtimes_\gamma G$. *Let us define the representation* Ψ *of* B *on* $\mathbb{C}^2 \otimes \mathbb{C}^2$ *by:*

$$\begin{aligned} \Psi(a) &= 1 \otimes a, \\ \Psi(U^\zeta) &= V \otimes U, \\ \Psi(U^\xi) &= U \otimes V. \end{aligned}$$

First, we observe that:

$$\begin{aligned} \Psi(U^\zeta)\Psi(a)\Psi(U^\zeta)^* &= 1 \otimes UaU^* = \Psi(\gamma_\zeta(a)), \\ \Psi(U^\xi)\Psi(a)\Psi(U^\xi)^* &= 1 \otimes VaV^* = \Psi(\gamma_\xi(a)). \end{aligned}$$

Therefore Ψ *is indeed defining a representation of* B. *Moreover:*

$$\Psi(U^g) = \Lambda'(g) \otimes \Lambda(g)$$

for $g \in G$. *Since* Λ' *is irreducible,* Ψ *is irreducible as well by Theorem (3.5). Last, the commutant of* $\Psi(A)$ *is* $M_2(\mathbb{C})$, *i.e. the restriction of* Ψ *to* A *is the direct sum of two copies of the identity representation of* A.

We now turn to the special case of cyclic groups where the representation Ψ of Theorem (3.4) is always minimal, i.e. its restriction to A is always an irreducible representation of A. We shall characterize such minimal representations in terms of the fixed point C*-subalgebra A_1 of A.

4. Actions of Finite Cyclic Groups

Let A be a unital C*-algebra and σ be a *-automorphism of A of period n, for $n \in \mathbb{N}$, i.e. $\sigma^n = \mathrm{Id}_A$. We shall not assume that n is the smallest such natural integer, i.e. σ may be of an order dividing n. The automorphism σ naturally generates an action of $\mathbb{Z}_n$ on A by letting $\alpha_z(a) = \sigma^k(a)$ for all $z \in \mathbb{Z}_n$ and $k \in \mathbb{Z}$ of class z modulo n. The crossed-product $A \rtimes_\alpha \mathbb{Z}_n$ will be simply denoted by $A \rtimes_\sigma \mathbb{Z}_n$, and the canonical unitary $U^1 \in A \rtimes_\sigma \mathbb{Z}_n$ corresponding to $1 \in \mathbb{Z}_n$ will simply be denoted by U. The C*-algebra $A \rtimes_\sigma \mathbb{Z}_n$ is universal among all C*-algebras generated by a copy of A and a unitary u such that $u^n = 1$ and $uau^* = \sigma(a)$.

Theorem (3.4) already provides much information about the structure of irreducible representations of $A \rtimes_\sigma \mathbb{Z}_n$. Yet we shall see it is

possible in this case to characterize these representations in terms of irreducible representations of A and of the fixed point C*-subalgebra A_1 of A for σ. Of central importance in this characterization are minimal representations of A for σ and their relation to irreducible representations of A_1. We start this section with the exploration of this connection. Next, we propose a full characterization of irreducible representations of $A \rtimes_\sigma \mathbb{Z}_n$.

4.1. **Minimal Representations.** An extreme case of irreducible representation for crossed-products is given by:

Definition 4.1. *Let Π be an irreducible representation of $A \rtimes_\alpha G$ is called minimal when its restriction to A is irreducible. Moreover, if π is an irreducible representation of A such that there exists some irreducible representation Π of $A \rtimes_\alpha G$ whose restriction to A is π, then we say that π is minimal for the action α of G.*

Such representations play a central role in the description of irreducible representations of $A \rtimes_\sigma \mathbb{Z}_n$ when σ is an automorphism of period n. We propose to characterize them in term of the fixed point C*-subalgebra A_1 of A. The set $\widehat{\mathbb{Z}_n}$ of irreducible representations of $\mathbb{Z}_n$ is the Pontryagin dual of $\mathbb{Z}_n$ which we naturally identify with the group $\mathbb{U}_n$ of n^{th} roots of the unit in $\mathbb{C}$. Let $\lambda \in \mathbb{U}_n$. Thus $k \in \mathbb{Z}_n \mapsto \lambda^n$ is an irreducible representation of $\mathbb{Z}_n$ and the spectral subspace A_λ of A for λ is given by $\{a : \sigma(a) = \lambda a\}$. Indeed, A_λ is by definition the range of the projection $P_\lambda : a \in A \mapsto \frac{1}{n}\sum_{k=0}^{n-1} \lambda^{-k}\sigma^k(a)$ by Equality (3.1), and it is easy to check that $P_\lambda(a) = a \iff \sigma(a) = \lambda a$ from the definition of P_λ.

Theorem 4.2. *Let σ be a *-automorphism of a unital C^*-algebra A of period n. Let Π be an irreducible representation of $A \rtimes_\sigma \mathbb{Z}_n$ on a Hilbert space $\mathcal{H}$ and let π_A be its restriction to A. Let Σ be the spectrum of $U_\Pi := \pi(U)$. Now, Σ is a subset of $\mathbb{U}_n$; let us write $\Sigma = \{\lambda_1, \ldots, \lambda_p\}$ and denote the spectral subspace of U_Π associated to λ_j by $\mathcal{H}_j$. With the decomposition $\mathcal{H} = \oplus_{k=1}^p \mathcal{H}_k$, we write, for all $a \in A$:*

$$\pi_A(a) = \begin{bmatrix} \alpha_{11}(a) & \alpha_{12}(a) & \cdots & \alpha_{1p}(a) \\ \alpha_{21}(a) & \alpha_{22}(a) & & \alpha_{2p}(a) \\ \vdots & & \ddots & \vdots \\ \alpha_{p1}(a) & \alpha_{p2}(a) & \cdots & \alpha_{nn}(a) \end{bmatrix}. \tag{4.1}$$

*Then for $k, j \in \{1, \ldots, p\}$ the map α_{jk} is a linear map on $A_{\lambda_j \overline{\lambda_k}}$ and null on $\oplus_{\mu \neq \lambda_j \overline{\lambda_k}} A_\mu$. Moreover, the maps α_{kk} are irreducible *-representations of the fixed point C^*-algebra A_1.*

Furthermore, the following are equivalent:

- *The representation π_A of A is irreducible, i.e. Π is minimal,*
- *The *-representations $\alpha_{11}, \ldots, \alpha_{pp}$ are pairwise not unitarily equivalent, i.e. for all $i \neq j \in \{1, \ldots, p\}$ the representation α_{ii} is not equivalent to α_{jj}.*

Proof. Since $U_\Pi^n = 1$, the spectrum of the unitary U_Π is a subset $\Sigma = \{\lambda_1, \ldots, \lambda_p\}$ of $\mathbb{U}_n$ for some $p \in \mathbb{N}$. We write $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_p$ where $\mathcal{H}_i$ is the spectral subspace of U_Π for the eigenvalue λ_i for $i = 1, \ldots, p$, so that $U_\Pi = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_p \end{bmatrix}$. Let $i, j \in \{1, \ldots, p\}$ and let α_{ij} be the map defined by Identity (4.1). First, it is immediate that α_{ij} is linear. Now, a simple computation shows that:

$$
\begin{aligned}
& U_\Pi \pi(a) U_\Pi^* = \\
= & \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_p \end{bmatrix} \begin{bmatrix} \alpha_{11}(a) & \cdots & \alpha_{1p}(a) \\ \vdots & & \vdots \\ \alpha_{p1}(a) & \cdots & \alpha_{pp}(a) \end{bmatrix} \begin{bmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_p} \end{bmatrix} \\
= & \begin{bmatrix} \alpha_{11}(a) & \lambda_1\overline{\lambda_2}\alpha_{12}(a) & \cdots & \lambda_1\overline{\lambda_p}\alpha_{1p}(a) \\ \lambda_2\overline{\lambda_1}\alpha_{21}(a) & \alpha_{22}(a) & & \lambda_2\overline{\lambda_p}\alpha_{2p}(a) \\ \vdots & & \ddots & \vdots \\ \lambda_p\overline{\lambda_1}\alpha_{p1}(a) & \lambda_p\overline{\lambda_2}\alpha_{p2}(a) & \cdots & \alpha_{pp}(a) \end{bmatrix} = \pi(\sigma(a)).
\end{aligned}
$$

Therefore for all $i, j \in \{1, \ldots, p\}$ we have that $\alpha_{ij}(\sigma(a)) = \lambda_i\overline{\lambda_j}\alpha_{ij}(a)$. Let $a \in A_\mu$ for $\mu \in \mathbb{U}_n$, i.e. $\sigma(a) = \mu a$. Then $\alpha_{ij}(\sigma(a)) = \mu\alpha_{ij}(a)$. Therefore either $\alpha_{ij}(a) = 0$ or $\mu = \lambda_i\overline{\lambda_j}$.

In particular, α_{jj} is a representation of A_1 for all $j \in \{1, \ldots, p\}$. Indeed, if $a \in A_1$ then $\alpha_{jk}(a) = 0$ if $j \neq k$ and thus $\pi_A(a)$ is diagonal. Since π_A is a representation of A, it follows from easy computations that α_{jj} are representations of A_1.

Now, since $A \oplus AU \oplus \ldots \oplus AU^{n-1} = A \rtimes_\sigma \mathbb{Z}_n$, every element of the range of Π is of the form $\oplus_{j=0}^{n-1} \pi_A(a_j) U_\Pi^j$ for $a_0, \ldots, a_{n-1} \in A$. Now, let $i \in \{1, \ldots, p\}$. We observe that the (i,i) entry of $\oplus_{j=0}^{n-1} \pi_A(a_i) U_\Pi^j$ in the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_p$ is given by $\sum_{j=0}^{n-1} \lambda_i^j \alpha_{ii}(a_j) = \alpha_{ii}\left(\sum_{j=0}^{n-1} \lambda_i^j a_j\right)$. Hence, the (i,i) entries of operators in the range of Π are exactly given by the operators in the range of α_{ii}. Now, let T be any operator acting on $\mathcal{H}$. Since Π is irreducible, by the Von Neumann double commutant Theorem [5, Theorem 1.7.1], T is the limit, in the weak operator topology (WOT), of elements in the range of Π. In

particular, the (i,i) entry of T in the decomposition $\mathcal{H}=\mathcal{H}_1\oplus\ldots\oplus\mathcal{H}_p$ is itself a WOT limit of elements in the range of α_{ii} since the left and right multiplications by a fixed operator are WOT continuous [5, p. 16]. Therefore, the range of α_{ii} is WOT dense in $\mathcal{H}_i$. Thus, by the double commutant theorem again, α_{ii} is irreducible.

We now turn to characterizing minimal representations. We first establish a necessary condition.

Suppose that there exists $i,j\in\{1,\ldots,p\}$ with $i\neq j$ and a unitary u such that $u\alpha_{ii}u^*=\alpha_{jj}$. In the decomposition $\mathcal{H}=\mathcal{H}_1\oplus\ldots\oplus\mathcal{H}_p$, define the block-diagonal unitary

$$D_u^i=\underbrace{1\oplus\ldots\oplus 1}_{i-1\text{ times}}\oplus u\oplus\underbrace{1\oplus\ldots\oplus 1}_{p-i\text{ times}}.$$

Then by conjugating π_A by D_u^i, we see that we may as well assume $\alpha_{ii}=\alpha_{jj}$. Yet, this implies that in the WOT-closure of the range of π_A, every operator has the same (i,i) and (j,j) entry in the decomposition $\mathcal{H}=\mathcal{H}_1\oplus\ldots\oplus\mathcal{H}_p$. Hence the range of π_A is not WOT-dense and thus π_A is reducible, so Π is not minimal.

We now prove that our necessary condition is also sufficient. Assume that $\alpha_{11},\ldots,\alpha_{pp}$ are pairwise not unitary equivalent. The claim is that π_A is irreducible.

Let $T\in(\pi(A))'$. Decompose $T=\begin{bmatrix}T_{11}&\cdots&T_{1p}\\ \vdots&&\vdots\\ T_{p1}&\cdots&T_{pp}\end{bmatrix}$ with respect to the decomposition $\mathcal{H}=\oplus_{i=1}^p\mathcal{H}_i$. Let $i\neq j$. First, note that if $a\in A_1$ then $\alpha_{ij}(a)=0$. Second, since T commutes with $\pi_A(a)$ for $a\in A_1$, we have:

$$\alpha_{ii}(a)\,T_{ij}=T_{ij}\alpha_{jj}(a)\ \text{ for } a\in A_1. \tag{4.2}$$

By Lemma (2.3), since α_{ii} and α_{jj} are irreducible and not unitarily equivalent for $i\neq j$, we conclude that $T_{ij}=0$. Moreover, for all $i\in\{1,\ldots,p\}$ and $a\in A_1$ we have $\alpha_{ii}(a)\,T_{ii}=T_{ii}\alpha_{ii}(a)$. Since α_{ii} is irreducible, we conclude that T_{ii} is a scalar. Therefore, the operator T commutes with the operator U_Π. Since Π is irreducible, we conclude that T itself is a scalar. Therefore, π_A is an irreducible representation of A and thus Π is minimal. □

Together with Theorem (3.4), Theorem (4.2) will allow us to now develop further the description of arbitrary irreducible representations of crossed-products by finite cyclic groups. It is interesting to look at a few very simple examples to get some intuition as to what could be a more complete structure theory for irreducible representations of

crossed-products by $\mathbb{Z}_n$. First of all, one should not expect in general that the spectrum of U_Π is a coset of $\mathbb{Z}_n$, as the simple action of $\sigma = \mathrm{Ad}\begin{bmatrix} i & \\ & e^{i\frac{3\pi}{4}} \end{bmatrix}$ on $M_2(\mathbb{C})$ shows. In this case, the identity is the only irreducible representation of the crossed-product $M_2(\mathbb{C}) \rtimes_\sigma \mathbb{Z}_4 = M_2(\mathbb{C})$ and clearly $\left\{i, e^{i\frac{3\pi}{4}}\right\}$ is not a coset of $\mathbb{Z}_4$. Of course, this is an example of a minimal representation.

In [4], we showed that all irreducible representations of $A \rtimes_\sigma \mathbb{Z}_2$ where regular or minimal. The following example shows that we can not expect the same in the general case.

Example 4.3. *Let $A = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and define $\sigma(M \oplus N) = WNW^* \oplus M$ with $W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\sigma^4 = \mathrm{Id}_A$ and $\sigma^2(M \oplus N) = WMW^* \oplus WNW^*$. Now, let $\pi_i : M_1 \oplus M_2 \in A \mapsto M_i$ with $i = 1, 2$. Of course, π_1, π_2 are the only two irreducible representations of A up to equivalence, and they are not equivalent to each other (since they have complementary kernels). Now, we consider the following representation Π of $A \rtimes_\sigma \mathbb{Z}_4$. It acts on $\mathbb{C}^4$. We set:*

$$\pi_A = \begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{bmatrix}$$

and:

$$U_\Pi = \begin{bmatrix} 0 & 1 \\ W & 0 \end{bmatrix}.$$

First, observe that Π thus defined is irreducible. Indeed, M commutes with π_A if and only if $M = \begin{bmatrix} \lambda & b \\ c & \mu \end{bmatrix}$ with $\lambda, \mu \in \mathbb{C}$ and $b\pi_2(a) = \pi_1(a)c$ with $a \in A$. Now, M commutes with U_Π if and only if $\lambda = \mu$ and $Wb = c$. Now, let $a \in M_2(\mathbb{C})$ be arbitrary; then $b\pi_2(a \oplus Wa) = \pi_1(a \oplus Wa)\, c$ i.e.

$$bWa = abW.$$

Hence bW is scalar. So $b = \lambda W$. Thus b commutes with W. But then for an arbitrary a we have $b\pi_2(aW \oplus a) = \pi_1(aW \oplus a)\, bW$ i.e. $ba = aWbW = ab$ so b commutes with $M_2(\mathbb{C})$ and thus is scalar. Hence $b = 0$. So $M = \lambda 1$ for $\lambda \in \mathbb{C}$ as needed.

Moreover, the restriction of Π to A is $\pi_A = \pi_1 \oplus \pi_2$. Thus, π_A is reducible. Now, the fixed point C^*-algebra A_1 is the C^*-algebra $\left\{M \oplus M : M = \begin{bmatrix} a & b \\ b & a \end{bmatrix}; a, b \in \mathbb{C}\right\}$. Thus, A_1 has two irreducible

representations which are not equivalent:

$$\varphi_1 : \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in A_1 \mapsto a + b$$

and

$$\varphi_2 : \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in A_1 \mapsto a - b.$$

We note that for $i = 1, 2$ we have π_i restricted to A_1 is $\varphi_1 \oplus \varphi_2$.

Now, using the notations of Example (4.3), Π is not regular, since the restriction of any irreducible regular representation to the fixed point algebra A_1 is given by the sum of several copies of the same irreducible representation of A_1. Trivially, Π is not minimal either since $\Pi_{|A} = \pi_1 \oplus \pi_2$. However, both π_1 and π_2 are minimal for the action of σ^2. Moreover, both π_1 and π_2 restricted to A_1 are the same representation $\alpha_1 \oplus \alpha_2$. We shall see in the next section that this pattern is in fact general.

4.2. **Characterization of Irreducible Representations.** We now present the main result of this paper concerning crossed products by finite cyclic groups. In this context, one can go further than Theorem (3.4) to obtain a characterization of irreducible representations of the crossed-products in term of the C*-algebras A and A_1. The next lemma is the sufficient condition for this characterization.

Lemma 4.4. *Let π_1 be an irreducible representation of A acting on a Hilbert space $\mathcal{J}$. Assume that there exists a unitary V on $\mathcal{J}$ such that for some $m, k \in \{1, \ldots, n\}$ with $n = mk$ we have $\pi_1 \circ \sigma^m = V \pi_1 V^*$ and $V^k = 1$, and that m is the smallest such nonzero natural integer, i.e. $\pi_1 \circ \sigma^j$ is not unitarily equivalent to π_1 for $j \in \{2, \ldots, m-1\}$. Then define the following operators on the Hilbert space $\mathcal{H} = \underbrace{\mathcal{J} \oplus \ldots \oplus \mathcal{J}}_{m \text{ times}}$:*

$$\Pi(U) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ V & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and for all $a \in A$:

$$\pi_A(a) = \begin{bmatrix} \pi_1(a) & & & \\ & \pi_1 \circ \sigma(a) & & \\ & & \ddots & \\ & & & \pi_1 \circ \sigma^{m-1}(a) \end{bmatrix}.$$

Then the unique extension of Π *to* $A \rtimes_\sigma \mathbb{Z}_n$ *is an irreducible representation of* $A \rtimes_\sigma \mathbb{Z}_n$*.*

Proof. An easy computation shows that Π thus defined is a representation of $A \rtimes_\sigma \mathbb{Z}_n$ on $\mathcal{H} = \underbrace{\mathcal{J} \oplus \ldots \oplus \mathcal{J}}_{m \text{ times}}$. Write $\pi_i = \pi_1 \circ \sigma^{i-1}$ for $i = 1, \ldots, m$. Let T be an operator which commutes with the range of Π. Then T commutes with $\pi_A := \Pi_{|A}$. Writing T in the decomposition $\mathcal{H} = \mathcal{J} \oplus \ldots \oplus \mathcal{J}$ as:

$$T = \begin{bmatrix} T_{11} & \cdots & T_{1m} \\ \vdots & & \vdots \\ T_{m1} & \cdots & T_{mm} \end{bmatrix}$$

Let $i, j \in \{1, \ldots, m\}$. Since $T\pi_A(a) = \pi_A(a)T$ for all $a \in A$, we conclude that $\pi_i(a)T_{ij} = T_{ij}\pi_j(a)$. By Lemma (2.3), since π_i and π_j are irreducible and not unitarily equivalent, we conclude that $T_{ij} = 0$. Moreover, T_{ii} commutes with π_i which is irreducible, so we conclude that:

$$T = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$$

for $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$. Since T commutes with U_Π we conclude that $\lambda_1 = \lambda_i$ for all $i \in \{1, \ldots, m\}$. Hence Π is irreducible. $\square$

We now are ready to describe in detail the structure of irreducible representations of crossed-products by finite cyclic groups in terms of irreducible representations of A and A_1.

Theorem 4.5. *Let* σ *be a *-automorphism of period* n *of a unital* C^**-algebra* A*. Then the following are equivalent:*

1. Π *is an irreducible representation of* $A \rtimes_\sigma \mathbb{Z}_n$*,*
2. *There exists* $k, m \in \mathbb{N}$ *with* $km = n$*, an irreducible representation* π_1 *of* A *on a Hilbert space* $\mathcal{J}$ *and a unitary* V *on* $\mathcal{J}$ *such that* $V^k = 1$ *and* $V\pi_1(\cdot)V = \pi_1 \circ \sigma^m(\cdot)$ *such that:*

$$\Pi(U) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ V & & & 0 \end{bmatrix}$$

and for all $a \in A$:

$$\Pi(a) = \begin{bmatrix} \pi_1(a) & & & \\ & \pi_1 \circ \sigma(a) & & \\ & & \ddots & \\ & & & \pi_1 \circ \sigma^{m-1}(a) \end{bmatrix}$$

where for any $i \in \{1, \ldots, m-1\}$ the representations π_1 and $\pi_1 \circ \sigma^i$ are not equivalent.

Moreover, if (2) holds then the representation ψ of $A \rtimes_{\sigma^m} \mathbb{Z}_k$ on $\mathcal{J}$ defined by $\psi(a) = \pi_1(a)$ for $a \in A$ and $\psi(U) = V$ is a minimal representation of $A \rtimes_{\sigma^m} \mathbb{Z}_k$. Let η be the cardinal of the spectrum of V. The restriction of π_1 to A_1 is therefore the sum of η irreducible representations $\varphi_1, \ldots, \varphi_\eta$ of A_1 which are not pairwise equivalent. Last, the restriction of $\pi_1 \circ \sigma^i$ to A_1 is unitarily equivalent to $\varphi_1 \oplus \ldots \oplus \varphi_\eta = \pi_{1|A_1}$ for all $i \in \{0, \ldots, m-1\}$.

Proof. By Lemma (4.4), (2) implies (1). We now turn to the proof of (1) implies (2). Let Π be an irreducible representation of $A \rtimes_\sigma \mathbb{Z}_n$. By Theorem (3.4), there exists $m \in \mathbb{N}$ such that m divides n, an irreducible representation π_1 of A on some space $\mathcal{H}_1$ and $r \in \mathbb{N}$ with $r > 0$ such that, if $\pi = r \cdot \pi_1$ then up to conjugating Π by some unitary:

- For all $i = 1, \ldots, m-1$ the representation $\pi \circ \sigma^i$ is not equivalent to π,
- The representation $\pi \circ \sigma^m$ is equivalent to π,
- We have the decomposition $\mathcal{H} = \mathcal{J}_0 \oplus \ldots \oplus \mathcal{J}_{m-1}$ where $\mathcal{J}_i$ is the space on which $(r \cdot \pi) \circ \sigma^i$ acts for $i \in \{0, \ldots, m\}$ and is isometrically isomorphic to $\mathcal{J}$,
- In the decomposition, $\mathcal{H} = \mathcal{J}_0 \oplus \ldots \oplus \mathcal{J}_{m-1}$ there exists unitaries $U_1, \ldots, U_m$ such that:

$$U_\Pi = \begin{bmatrix} 0 & U_1 & 0 & \cdots & 0 \\ 0 & 0 & U_2 & 0 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & U_{m-1} \\ U_m & 0 & \cdots & 0 & 0 \end{bmatrix}$$

with $(\pi_i \circ \sigma) U_i = U_i \pi_{i+1}$ and $U_i : \mathcal{H}_{i+1} \longrightarrow \mathcal{H}_i$ for all $i \in \mathbb{Z}_m$.

Indeed, if $G = \mathbb{Z}_n$ in Theorem (3.4) then H, as a subgroup of G, is of the form $(m\mathbb{Z})/n\mathbb{Z}$ with m dividing n, and if we let $g_1 = 0$, $g_2 = 1, \ldots, g_m = m-1$ then we can check that this choice satisfies the hypothesis of Theorem (3.4). With this choice, the permutation σ^1 is then easily seen to be given by the cycle $(1\ 2\ \ldots\ m)$.

We will find it convenient to introduce some notation for the rest of the proof. By Theorem (3.4), for $i \in \{0, \ldots, m-1\}$, after possibly conjugating Π by some unitary, we can decompose $\mathcal{J}_i$ as $\mathcal{H}_{ri+1} \oplus \ldots \oplus \mathcal{H}_{r(i+1)}$, where $\mathcal{H}_{ri+j}$ is isometrically isomorphic to $\mathcal{H}_1$ for all $j \in \{1, \ldots, r\}$, so that the restriction of $\Pi_{|A}$ to the space $\mathcal{J}_i$ is written $\left(\underbrace{\pi_1 \oplus \ldots \oplus \pi_1}_{r \text{ times}}\right) \circ \sigma^i$ in this decomposition.

We now show how to conjugate Π by a unitary to simplify its expression further.

If we define the unitary Υ from $\mathcal{H} = \mathcal{J}_0 \oplus \ldots \oplus \mathcal{J}_{m-1}$ onto $\oplus_1^m \mathcal{J}_{m-1}$ by:

$$\Upsilon = \begin{bmatrix} U_m^* U_{m-1}^* \cdots U_1^* & & & \\ & U_m^* \cdots U_2^* & & \\ & & \ddots & \\ & & & U_m^* \end{bmatrix}$$

then the unitary $\mathrm{Ad}\,(\Upsilon) \circ \Pi\,(U)$ of $\oplus_1^m \mathcal{J}_{m-1}$ is of the simpler form

$$\mathrm{Ad}\,\Upsilon \circ \Pi\,(U) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ V & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{4.3}$$

for some unitary V of $\mathcal{J}_{m-1}$. Moreover, if we write $\rho_1 = \mathrm{Ad}\,(U_i^* \ldots U_1^*) \circ \pi_1$, then:

$$\mathrm{Ad}\,\Upsilon \circ \pi_A = \bigoplus_{j=1}^{m} \left(\underbrace{\rho_1 \circ \sigma^{j-1} \oplus \ldots \oplus \rho_1 \circ \sigma^{j-1}}_{r \text{ times}} \right)$$

and ρ_1 is by definition an irreducible representation of A unitarily equivalent to π_1.

To simplify notations, we shall henceforth drop the notation $\mathrm{Ad}\,\Upsilon$ and simply write Π for $\mathrm{Ad}\,\Upsilon \circ \Pi$. In other words, we replace Π by $\mathrm{Ad}\,\Upsilon \circ \Pi$ and we shall use the notations introduced to study Π henceforth, with the understanding that for all $j = 0, \ldots, m-1$ and $k = 1, \ldots, r$ we have that $\pi_{rj+k} = \pi_1 \circ \sigma^j$, that $\mathcal{J}_j$ is an isometric copy of $\mathcal{J}_0$ and that $\mathcal{H} = \mathcal{J}_0 \oplus \ldots \oplus \mathcal{J}_{m-1}$ with $\mathcal{J}_j = \mathcal{H}_{rj+1} \oplus \ldots \oplus \mathcal{H}_{r(j+1)}$ where π_{rj+k} acts on $\mathcal{H}_{rj+k}$ which is an isometric copy of $\mathcal{H}_1$. Moreover, U_Π is given by Equality (4.3) for some unitary V of $\mathcal{J}_0$.

We are left to show that each irreducible subrepresentation of π_A is of multiplicity one, i.e. $r = 1$. We recall that we have shown above

that $H = (m\mathbb{Z})/n\mathbb{Z}$ with $n = mk$ and $k \in \mathbb{N}$. Using the notations of Theorem (3.4), the representation Ψ defined by $\Psi(a) = \pi(a)$ for all $a \in A$ and $\Psi(U^m) = V$ is an irreducible representation of $A \rtimes_\alpha H$. Now $A \rtimes_\alpha H$ is *-isomorphic to $A \rtimes_{\alpha^m} \mathbb{Z}_k$ by universality of the C*-crossed-product, and we now identify these two C*-algebras. The image of $U^m \in A \rtimes_\alpha H$ in the crossed-product $A \rtimes_{\alpha^m} \mathbb{Z}_k$ is denoted by υ and is the canonical unitary of $A \rtimes_{\alpha^m} \mathbb{Z}_k$. Thus by Theorem (3.4) Ψ is an irreducible representation of $A \rtimes_{\alpha^m} \mathbb{Z}_k$ which (up to conjugacy) acts on the space $\mathbb{C}^r \otimes \mathcal{H}_1$ and is of the form $\Psi(a) = 1_{\mathbb{C}^r} \otimes \pi_1(a)$ for $a \in A$ and $\Psi(\upsilon^z) = \Omega(z) \otimes W(z)$ for $z \in \mathbb{Z}_k$ where Ω and W are some unitary projective representations of $\mathbb{Z}_k$ on $\mathbb{C}^r$ and $\mathcal{H}_1$ respectively, with Ω being irreducible. Since $\mathbb{Z}_k$ is cyclic, the range of the projective representation Ω is contained in the C*-algebra $C^*(\Omega(1))$ which is Abelian since $\Omega(1)$ is a unitary. Hence, since Ω is irreducible, $C^*(\Omega(1))$ is an irreducible Abelian C*-algebra of operators acting on $\mathbb{C}^r$. Hence $r = 1$ and $\mathcal{J} = \mathcal{H}_1$. Moreover, since $U_\Pi^m = V \oplus \ldots \oplus V$ then $U_\Pi^n = V^k \oplus \ldots \oplus V^k = 1_\mathcal{H}$ and thus $V^k = 1_\mathcal{J}$. Therefore, (2) holds as claimed.

Last, we also observed that $V\pi_1 V = \pi_1 \circ \sigma^k$ by construction (since $U_\Pi^k = V \oplus \ldots \oplus V$). Hence by definition, since π_1 is irreducible, the representation ψ of $A \rtimes_{\sigma^k} \mathbb{Z}_\mu$ defined by $\psi(a) = \pi_1(a)$ for $a \in A$ and $\psi(U) = V$ is minimal. Hence, by Theorem (4.2), the restriction of π_1 to the fixed point C*-algebra A_1 is the direct sum of η irreducible representations $\varphi_1, \ldots, \varphi_\eta$ of A_1 such that φ_i and φ_j are not unitarily equivalent for $i \neq j \in \{1, \ldots, \eta\}$, where η is the cardinal of the spectrum of V. Moreover, since $\pi_i = \pi_1 \circ \sigma^i$ it is immediate that π_i restricted to A_1 equals to π_1 restricted to A_1. This concludes our proof. □

Corollary 4.6. *Let Π be an irreducible representation of $A \rtimes_\sigma \mathbb{Z}_n$. The following are equivalent:*

(1) *Up to unitary equivalence, Π is an irreducible regular representation of $A \rtimes_\sigma \mathbb{Z}_n$, i.e. it is induced by a unique irreducible representation π of A and:*

$$U_\Pi = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}$$

while $\pi_A = \oplus_{i=0}^{n-1} \pi \circ \sigma^i$ and $\pi \circ \sigma^i$ is not equivalent to $\pi \circ \sigma^j$ for $i, j = 1, \ldots, n-1$ with $i \neq j$,

(2) *There exists an irreducible subrepresentation* π *of* $\Pi_{|A}$ *such that* $\pi \circ \sigma^i$ *is not equivalent to* π *for* $i = 1, \ldots, n-1$,
(3) *There exists a unique irreducible representation* φ *of* A_1 *such that* $\Pi_{|A_1}$ *is equivalent to* $n \cdot \varphi$,
(4) *There is no* $k \in \{1, \ldots, n-1\}$ *such that the* C^**-algebra generated by* $\Pi(A)$ *and* U_Π^k *is reducible.*

Proof. It is a direct application of Theorem (4.5). □

We thus have concluded that all irreducible representations of crossed products by finite cyclic groups have a structure which is a composite of the two cases found in [4]. Indeed, such representations cycle through a collection of minimal representations, which all share the same restriction to the fixed point algebra. The later is a finite sum of irreducible mutually disjoint representations of the fixed point algebra.

Remark 4.7. *Let* σ *be an order* n *automorphism of a unital* C^**-algebra* A *and let* Π *be an irreducible representation of* $A \rtimes_\sigma \mathbb{Z}$. *We recall* [12] *that* $A \rtimes_\sigma \mathbb{Z}$ *is generated by* A *and a unitary* U *such that* $UaU^* = \sigma(a)$ *for all* $a \in A$ *and is universal for these commutation relations. We denote* $\Pi(U)$ *by* U_Π *and* $\Pi(a)$ *by* $\pi(a)$ *for all* $a \in A$. *Now, note that* U_Π^n *commutes with* π *since* $\sigma^n = \mathrm{Id}_A$ *and of course* U_Π^n *commutes with* U_Π *so, since* Π *is irreducible, there exists* $\lambda \in \mathbb{T}$ *such that* $U_\Pi^n = \lambda$. *Now, define* $V_\Pi = \overline{\mu} U_\Pi$ *for any* $\mu \in \mathbb{T}$ *such that* $\mu^n = \lambda$. *Then* $V_\Pi^n = 1$ *and thus* (π, V_Π) *is an irreducible representation of* $A \rtimes_\sigma \mathbb{Z}_n$ *which is then fully described by Theorem (4.5).*

In the last section of this paper, we give a necessary condition on irreducible representations of crossed-products by the group $\mathfrak{S}_3$ of permutations of $\{1, 2, 3\}$. This last example illustrates some of the behavior which distinguish the conclusion of Theorem (3.4) from the one of Theorem (4.5).

5. Application: Crossed-Products by the permutation group on $\{1, 2, 3\}$

As an application, we derive the structure of the irreducible representations of crossed-products by the group $\mathfrak{S}_3$ of permutations of $\{1, 2, 3\}$. This group is isomorphic to $\mathbb{Z}_3 \rtimes_\gamma \mathbb{Z}_2$ where γ is defined as follows: if η and τ are the respective images of $1 \in \mathbb{Z}$ in the groups $\mathbb{Z}_3$ and $\mathbb{Z}_2$ then the action γ of $\mathbb{Z}_2$ on $\mathbb{Z}_3$ is given by $\gamma_\tau(\eta) = \eta^2$. Thus in $\mathbb{Z}_3 \rtimes_\gamma \mathbb{Z}_2$ we have $\tau\eta\tau = \eta^2$, $\tau^2 = 1$ and $\eta^3 = 1$ (using the multiplicative notation for the group law). An isomorphism between $\mathfrak{S}_3$ and $\mathbb{Z}_3 \rtimes_\gamma \mathbb{Z}_2$ is given by sending the transposition $(1\ 2)$ to τ and the 3-cycle $(1\ 2\ 3)$

to η. From now on we shall identify these two groups implicitly using this isomorphism.

Theorem 5.1. *Let α be an action of $\mathfrak{S}_3$ on A. Let Π be an irreducible representation of $A \rtimes_\alpha \mathfrak{S}_3$. We denote by τ and η the permutations* $(1\ 2)$ *and* $(1\ 2\ 3)$*. The set $\{\tau, \eta\}$ is a generator set of $\mathfrak{S}_3$. We denote by U_τ and U_η the canonical unitaries in $A \rtimes_\alpha \mathfrak{S}_3$ corresponding respectively to τ and η. Then either (up to a unitary conjugation of Π):*

- *Π is minimal, i.e. $\Pi_{|A}$ is irreducible,*
- *There exists an irreducible representation π_1 on $\mathcal{H}_1$ of A such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ with $\pi_A = \pi_1 \oplus \pi_1 \circ \alpha_\tau$. Then $\Pi(U_\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in this decomposition.* Observe that π_1 may or not be equivalent to $\pi_1 \circ \alpha_\tau$. *Moreover, π_1 and $\pi_1 \circ \alpha_\tau$ are minimal for the action of η.*
- *There exists an irreducible representation π_1 on $\mathcal{H}_1$ of A such that π_1 and $\pi_1 \circ \alpha_{\eta^i}$ are non equivalent for $i = 1, 2$ and such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_1$ with $\pi_A = \pi_1 \oplus \pi_1 \circ \alpha_\eta \oplus \pi_1 \circ \alpha_{\eta^2}$. Then $\Pi(U_\eta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ in this decomposition.*
- *Last, there exists an irreducible representation π_1 on $\mathcal{H}_1$ of A such that $\pi_1 \circ \alpha_\sigma$ is not equivalent to π_1 for $\sigma \in \mathfrak{S}_3 \setminus \{\mathrm{Id}\}$ and $\mathcal{H} = \mathcal{H}_1^{\oplus 6}$ with:*

$$\pi_A = \pi_1 \oplus \pi_1 \circ \alpha_\eta \oplus \pi_1 \circ \alpha_{\eta^2} \oplus \pi_1 \circ \alpha_\tau \oplus \pi_1 \circ \alpha_{\eta\tau} \oplus \pi_1 \circ \alpha_{\eta^2\tau}$$

and

$$\Pi(U_\eta) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

while

$$\Pi(U_\tau) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Proof. The C*-algebra $A \rtimes_\alpha \mathfrak{S}_3$ is generated by a copy of A and two unitaries U_τ and U_η that satisfy $U_\tau^2 = U_\eta^3 = 1$, $U_\tau U_\eta U_\tau = U_\eta^2$ and

for all $a \in A$ we have $U_\tau a U_\tau^* = \alpha_\tau(a)$ and $U_\eta a U_\eta^* = \alpha_\eta(a)$. Notice that $\mathfrak{S}_3 = \mathbb{Z}_3 \rtimes_\gamma \mathbb{Z}_2$ with $\gamma_\tau(\eta) = \tau\eta\tau$. So we have $A \rtimes_\alpha \mathfrak{S}_3 = \left(A \rtimes_{\alpha_\eta} \mathbb{Z}_3\right) \rtimes_\beta \mathbb{Z}_2$ where $\beta : a \in A \mapsto \alpha_\tau(a)$ and $\beta(U_\eta) = U_{\tau\eta\tau} = U_\eta^2$. Since $A \rtimes_{\alpha_\eta} \mathbb{Z}_3 = A + AU_\eta + AU_\eta^2$, the relation between β and α_η is given by:

$$\beta\left(x_1 + x_2 U_\eta + x_3 U_\eta^2\right) = \alpha_\tau(x_1) + \alpha_\tau(x_3) U_\eta + \alpha_\tau(x_2) U_\eta^2$$

for all x_1,x_2 and $x_3 \in A$. We now proceed with a careful analysis of β and α_η to describe all irreducible representations of $A \rtimes_\alpha \mathfrak{S}_3$.

Let Π be an irreducible representation of $A \rtimes_\alpha \mathfrak{S}_3$ on some Hilbert space $\mathcal{H}$. Thus Π is an irreducible representation of $\left[A \rtimes_{\alpha_\eta} \mathbb{Z}_3\right] \rtimes_\beta \mathbb{Z}_2$. We now have two cases: either $\Pi_{|A \rtimes_{\alpha_\eta} \mathbb{Z}_3}$ is irreducible or it is reducible.

Case 1: $\Pi_{|A \rtimes_{\alpha_\eta} \mathbb{Z}_3}$ is irreducible.: Hence Π is minimal for the action β of $\mathbb{Z}_2$. This case splits in two cases.

Case 1a: π_A is irreducible: Then Π is minimal for the action α of $\mathfrak{S}_3$ by definition.

Case 1b: π_A is reducible: By Theorem (4.5), there exists an irreducible representation π_1 of A on some Hilbert space $\mathcal{H}_1$ such that π_1, $\pi_1 \circ \alpha_\eta$ and $\pi_1 \circ \alpha_\eta^2$ are not unitarily equivalent, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_1$ and:

$$\Pi(U_\eta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \Pi(a) = \begin{bmatrix} \pi_1(a) & & \\ & \pi_1 \circ \alpha_\eta(a) & \\ & & \pi_1 \circ \alpha_{\eta^2}(a) \end{bmatrix}.$$

Case 2: $\Pi_{|A \rtimes_\beta \mathbb{Z}_3}$ is reducible.: From Theorem (4.5), or alternatively [4], there exists an irreducible representation π_1 of $A \rtimes_{\alpha_\eta} \mathbb{Z}_3$ such that for all $z \in A \rtimes_{\alpha_\eta} \mathbb{Z}_3$ we have:

$$\Pi(a) = \begin{bmatrix} \pi_1(z) & 0 \\ 0 & \pi_1 \circ \beta(z) \end{bmatrix} \text{ and } \Pi(U_\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5.1}$$

where π_1 and $\pi_1 \circ \beta$ are not unitarily equivalent.

This case splits again in two cases:

Case 2a: $\pi_{1|A}$ is irreducible: Thus π_1 is a minimal representation of $A \rtimes_{\alpha_\eta} \mathbb{Z}_3$. In particular:

$$\pi_A(a) = \begin{bmatrix} \pi_1(a) & 0 \\ 0 & \pi_1 \circ \alpha_\tau(a) \end{bmatrix}$$

and $\Pi(U_\eta)$ is a block-diagonal unitary in this decomposition. However, we can not conclude that $\pi_{1|A}$ and $\pi_{1|A} \circ \alpha_\tau$ are equivalent or non-equivalent. Examples (5.4) and (5.6) illustrate that both possibilities occur.

Case 2b: $\pi_{1|A}$ is reducible: Then $\Pi_{|A\rtimes_{\alpha_\eta}\mathbb{Z}_3}$ is described by Theorem (4.5). Since 3 is prime, only one possibility occurs: there exists an irreducible representation π of A such that $\pi\circ\alpha_\eta$ and $\pi\circ\alpha_\eta^2$ are not equivalent and:

$$\Pi(a)=\left[\begin{array}{ccc}\pi(a) & 0 & 0\\ 0 & \pi(\alpha_\eta(a)) & 0\\ 0 & 0 & \pi\left(\alpha_{\eta^2}(a)\right)\end{array}\right]$$

and $\Pi\left(U_\eta\right)=\left[\begin{array}{ccc}0&1&0\\0&0&1\\1&0&0\end{array}\right]$. Note that:

$$\Pi\left(\beta(U_\Pi)\right)=\left[\begin{array}{ccc}0&0&1\\1&0&0\\0&1&0\end{array}\right].$$

Together with (5.1), we get that $\mathcal{H}$ splits into the direct sum of six copies of the Hilbert space on which π acts and:

$$\Pi(a)=\left[\begin{array}{cccccc}\pi(a) &&&&&\\ & \pi(\alpha_\eta(a)) &&&&\\ && \pi(\alpha_{\eta^2}(a)) &&&\\ &&& \pi\left(\alpha_\tau(a)\right) &&\\ &&&& \pi\left(\alpha_{\eta\tau}(a)\right) &\\ &&&&& \pi\left(\alpha_{\eta^2\tau(a)}\right)\end{array}\right]$$

and

$$\Pi(U_\eta)=\left[\begin{array}{cccccc}0&1&0&0&0&0\\0&0&1&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&1&0&0\\0&0&0&0&1&0\end{array}\right]$$

while

$$\Pi\left(U_\tau\right)=\left[\begin{array}{cccccc}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{array}\right].$$

Thus Π is regular induced by π, and therefore, as Π is irreducible, $\pi\circ\alpha_\sigma$ is not equivalent to π for any $\sigma\in\mathfrak{S}_3\backslash\{\mathrm{Id}\}$ by Theorem (2.4).

This concludes our proof. □

We show that all four possibilities above do occur in a nontrivial manner. We use the generators τ and η as defined in Theorem (5.1). Denote by e the identity of $\{1,2,3\}$. Notice that $\tau^2 = \eta^3 = e$ and $\tau\eta\tau = \eta^2$ and $\tau\eta^2\tau = \tau$, while:

$$\mathfrak{S}_3 = \left\{ e, \eta, \eta^2, \tau, \eta\tau, \eta^2\tau \right\}.$$

In particular, $\{1, \eta, \eta^2\}$ is a normal subgroup of $\mathfrak{S}_3$. Now, consider the universal C*-algebra of the free group on three generators $A = C^*(\mathbb{F}_3)$ and denote by U_1, U_2 and U_3 its three canonical unitary generators. Then we define the action α of $\mathfrak{S}_3$ on A by setting $\alpha_\sigma(U_i) = U_{\sigma(i)}$ for any $\sigma \in \mathfrak{S}_3$. We now show that this simple example admits in a nontrivial way all types of representations described in Theorem (5.1).

Example 5.2. *There exists a nontrivial irreducible representation $\pi : C^*(\mathbb{F}_3) \to M_2(\mathbb{C})$ such that π and $\pi \circ \alpha_\tau$ are unitarily equivalent, but π and $\pi \circ \alpha_\eta$ are not. Indeed, set:*

$$\pi(U_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \pi(U_2) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad \pi(U_3) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We check easily that π is an irreducible $*$-representation. Since

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} [\pi \circ \alpha_\tau] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \pi,$$

π and $\pi \circ \alpha_\tau$ are unitarily equivalent. To see that π and $\pi \circ \alpha_\eta$ are not unitarily equivalent, notice that $\pi(U_1U_2 - U_2U_1) = 0$ but that:

$$\pi(U_2U_3 - U_3U_2) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

Example 5.3. *There exists a non trivial irreducible representation $\pi : C^*(\mathbb{F}_3) \to M_3(\mathbb{C})$ such that π and $\pi \circ \alpha_\tau$ are unitarily equivalent and π and $\pi \circ \alpha_\eta$ are also unitarily equivalent. Let $\lambda = \exp\left(\frac{1}{3}2i\pi\right)$. Define*

$$\pi(U_1) = \begin{bmatrix} 0 & \lambda \\ \lambda^2 & 0 \end{bmatrix}, \quad \pi(U_2) = \begin{bmatrix} 0 & \lambda^2 \\ \lambda & 0 \end{bmatrix} \text{ and } \pi(U_3) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let $V = \begin{bmatrix} 1 & 0 \\ 0 & \lambda^2 \end{bmatrix}$. We check that $V\pi(U_i)V^* = \pi\left(U_{(i+1) \bmod 3}\right)$. Then let $W = \pi(U_3)$. Then $W\pi(U_1)W^* = \pi(U_2)$, $W\pi(U_2)W^* = \pi(U_1)$, and $W\pi(U_3)W^* = \pi(U_3)$. Thus π is a minimal representation of $C^*(\mathbb{F}_3)$ for the action α of $\mathfrak{S}_3$.

Example 5.4. *There exists an irreducible representation $\pi : C^*(\mathbb{F}_3) \to M_3(\mathbb{C})$ such that π and $\pi \circ \alpha_\eta$ are unitarily equivalent, but π and $\pi \circ \alpha_\tau$ are not: Let $\lambda = \exp\left(\frac{1}{3}2\pi i\right)$ and define unitaries T and V by*

$$T = \begin{bmatrix} 0 & -\frac{4}{5} & -\frac{3}{5} \\ \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ \frac{3}{5} & \frac{12}{25} & -\frac{16}{25} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}$$

Define

$$\pi(U_1) = VTV^2 \qquad \pi(U_2) = V^2TV \qquad \pi(U_3) = T.$$

It is clear that π and $\pi \circ \alpha_\eta$ are unitarily equivalent. We will show that π and $\pi \circ \alpha_\tau$ are not unitarily equivalent. Suppose on the contrary that they are. Then there exists a unitary W such that $W = W^* = W^{-1}$ and

$$WTW = T, \qquad W\left(VTV^2\right)W = V^2TV \qquad W\left(V^2TV\right)W = VTV^2.$$

From here we conclude that VWV performs the same transformations, that is

$$\begin{aligned} (VWV)\,T\,(VWV)^* &= T, \\ (VWV)\left[VTV^2\right](VWV)^* &= V^2TV, \\ (VWV)\left[V^2TV\right](VWV)^* &= VTV^2. \end{aligned}$$

Indeed,

$$\begin{aligned} W\left(VTV^2\right)W &= V^2TV \text{ so} \\ V\left[W\left(VTV^2\right)W\right] &= V\left[V^2TV\right] = TV. \end{aligned}$$

Then we multiply both sides by V^2 from the right to get

$$VWVTV^2WV^2 = T.$$

Since

$$(VWV)^* = V^*W^*V^* = V^2WV^2,$$

we get the first equation. Similarly we get the other two.

Since π is irreducible we conclude that there exists a constant c such that

$$VWV = cW.$$

V has a precise form and when we compute $VWV - cW$ we conclude that this equation has a non zero solution iff $c = 1$, $c = \lambda$, or $c = \lambda^2$.

Moreover, the solutions have the form:

$$\begin{aligned} W &= \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & y \\ 0 & z & 0 \end{bmatrix} \quad \text{if } c = 1 \\ W &= \begin{bmatrix} 0 & x & 0 \\ y & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \quad \text{if } c = \lambda \\ W &= \begin{bmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{bmatrix} \quad \text{if } c = \lambda^2 \end{aligned}$$

for some $x, y, c \in \mathbb{C}$.

Now we easily check that T *does not commute with any of the three* W*'s. For example,*

$$\begin{bmatrix} x & 0 & 0 \\ 0 & 0 & y \\ 0 & z & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{4}{5} & -\frac{3}{5} \\ \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ \frac{3}{5} & \frac{12}{25} & -\frac{16}{25} \end{bmatrix} - \begin{bmatrix} 0 & -\frac{4}{5} & -\frac{3}{5} \\ \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ \frac{3}{5} & \frac{12}{25} & -\frac{16}{25} \end{bmatrix} \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & y \\ 0 & z & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \frac{3}{5}z - \frac{4}{5}x & \frac{4}{5}y - \frac{3}{5}x \\ \frac{3}{5}y - \frac{4}{5}x & \frac{12}{25}y - \frac{12}{25}z & -\frac{7}{25}y \\ \frac{4}{5}z - \frac{3}{5}x & \frac{7}{25}z & \frac{12}{25}z - \frac{12}{25}y \end{bmatrix}.$$

This of course implies that $x = y = z = 0$.

Example 5.5. *This example acts on* $A = C(\mathbb{T}^3)$. *Define for* $f \in C(\mathbb{T}^3)$ *and* $(z_1, z_2, z_3) \in \mathbb{T}^3$*:*

$$\alpha_\eta(f)(z_1, z_2, z_3) = f(z_2, z_3, z_1)$$

and

$$\alpha_\tau(f)(z_1, z_2, z_3) = f(z_2, z_1, z_3)$$

on $C(\mathbb{T}^3)$. *We can build a non trivial irreducible representation* $\pi : C(\mathbb{T}^3) \rightarrow \mathbb{C}$ *such that* π *and* $\pi \circ \alpha_\eta$ *are not unitarily equivalent and* π *and* $\pi \circ \alpha_\tau$ *are also not unitarily equivalent. Let* $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ *be such that* $x_1 \neq x_2$, $x_2 \neq x_3$, *and* $x_3 \neq x_1$.*Define* $\pi(f) = f(x)$. *Then we obtain an irreducible representation of the required type as the regular representation induced by* π, *using Theorem (2.4).*

Now, Theorem (3.4) allowed for the irreducible subrepresentations of $\Pi_{|A}$ to have multiplicity greater than one, for irreducible representations Π of $A \rtimes_\alpha G$. This situation is however prohibited when G is finite cyclic by Theorem (4.5). We show that finite polycyclic groups such as $\mathfrak{S}_3$ can provide examples where $\Pi_{|A}$ may not be multiplicity

free, thus showing again that Theorem (3.4) can not be strengthened to the conclusion of Theorem (4.5).

Example 5.6. *We shall use the notations of Theorem (5.1). There exists a unital C^*-algebra A, an action α of $\mathfrak{S}_3$ on A and an irreducible representation $\widetilde{\Pi} : A \rtimes_\alpha S_3 \to B(\mathcal{H} \oplus \mathcal{H})$ such that for all $x \in A$ we have:*

$$\widetilde{\Pi}(x) = \begin{bmatrix} \pi(x) & 0 \\ 0 & \pi(\alpha_\tau(x)) \end{bmatrix} \tag{5.2}$$

for some irreducible representation $\pi : A \to B(\mathcal{H})$ such that π and $\pi \circ \alpha_\tau$ are equivalent. Note that π is thus minimal for the action of α_η.

Indeed, let us start with any unital C^*-algebra A for which there exists an action α of $\mathfrak{S}_3$ and an irreducible representation $\Pi : A \rtimes_\alpha \mathfrak{S}_3 \to B(\mathcal{H})$ such that $\pi = \Pi_{|A}$ is also irreducible, i.e. Π is minimal. For instance, Example (5.3) provides such a situation. Let $V_\eta = \Pi(U_\eta)$ and $V_\tau = \Pi(U_\tau)$. Then for all $x \in A$

$$\begin{aligned} V_\eta \pi(x) V_\eta^* &= \pi(\alpha_\eta(x)), \\ V_\eta \pi(x) V_\eta^* &= \pi(\alpha_\eta(x)), \\ V_\tau^2 = 1,\ V_\eta^3 &= 1 \text{ and } V_\tau V_\eta V_\tau = V_\eta^2. \end{aligned}$$

Let $\omega = \exp\left(\frac{1}{3} 2\pi i\right)$. For $x \in A$ define $\widetilde{\Pi}(x)$ by (5.2); let $W_\eta = \widetilde{\Pi}(U_\eta)$ and $W_\tau = \widetilde{\Pi}(U_\tau)$ given by:

$$W_\eta = \begin{bmatrix} \omega V_\eta & 0 \\ 0 & \omega^2 V_\eta \end{bmatrix} \qquad W_\tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We easily check that:

$$W_\eta \widetilde{\Pi}(x) W_\eta^* = \widetilde{\Pi}(\alpha_\eta(x)),\ W_\tau \widetilde{\Pi}(x) W_\tau^* = \widetilde{\Pi}(\alpha_\tau(x)),$$

and:

$$(W_\tau)^3 = 1, (W_\tau)^2 = 1.$$

Moreover,

$$\begin{aligned} W_\tau W_\eta W_\tau &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \omega V_\eta & 0 \\ 0 & \omega^2 V_\eta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \omega^2 (V_\eta)^2 & 0 \\ 0 & \omega (V_\eta)^2 \end{bmatrix} = (V_\eta)^2, \end{aligned}$$

because $\omega^4 = \omega$.

We need to prove that $\widetilde{\Pi} : A \rtimes_\alpha S_3 \to B(\mathcal{H} \oplus \mathcal{H})$ is irreducible. Let

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be in the commutant of $\widetilde{\Pi}\left(A \rtimes_\alpha S_3\right)$. *For every* $x \in A$:

$$T\begin{bmatrix} \pi(x) & 0 \\ 0 & \pi(\alpha_\tau(x)) \end{bmatrix} = \begin{bmatrix} \pi(x) & 0 \\ 0 & \pi\left(\alpha_\tau(x)\right) \end{bmatrix} T.$$

Since π *is an irreducible representation of* A *and* $\pi \circ \alpha_\tau = V_\tau \pi V_\tau$ *by construction, we conclude by Lemma (2.3) that* a *and* b *are multiple of the identity, while* c *and* d *are multiples of* V_τ. *Since* $TW_\tau = W_\tau T$ *we conclude that* $a = d$ *and* $b = c$. *This means that*

$$T - aI = \begin{bmatrix} 0 & bV_\tau \\ bV_\tau & 0 \end{bmatrix}$$

is in the commutant of the $\widetilde{\pi}\left(A \rtimes_\alpha S_3\right)$. *However, this element must commute with* W_η. *This can only happen if* $b = 0$. *This completes the proof.*

Thus, using Example (5.6), there exists an irreducible representation $\widetilde{\Pi}$ of $C^*(\mathbb{F}_3) \rtimes_\alpha \mathfrak{S}_3$ such that $\widetilde{\Pi}_{|C^*(\mathbb{F}_3)}$ is the sum of two equivalent irreducible representations of $C^*(\mathbb{F}_3)$, a situation which is impossible for crossed-product by finite cyclic groups by Theorem (4.5).

In general, repeated applications of Theorem (4.5) can lead to detailed descriptions of irreducible representations of crossed-products of unital C*-algebra by finite polycyclic groups, based upon the same method as we used in Theorem (5.1). Of course, in these situations Theorem (3.4) provides already a detailed necessary condition on such representations, and much of the structure can be read from this result.

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